7.3 The Eigenvalue Method for Systems

$\vec{x}' = A\vec{x}$ homogeneous system, A is a constant matrix

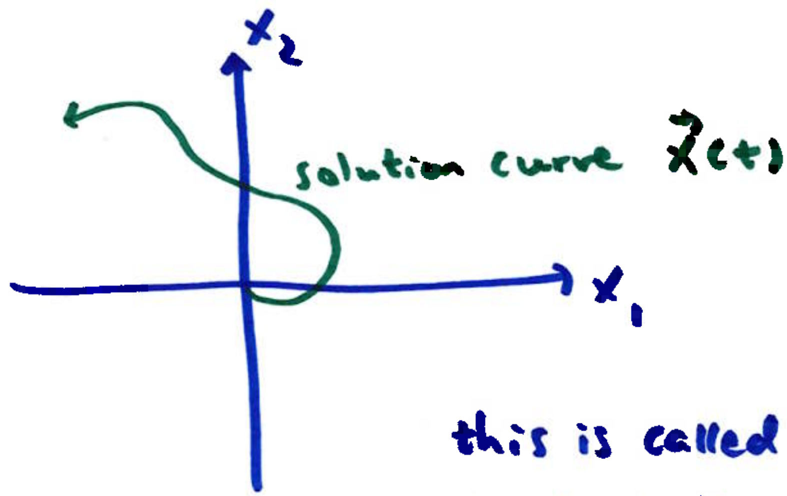
first, a qualitative way to understand solutions

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Solution: $x_1(t) =$

$x_2(t) =$

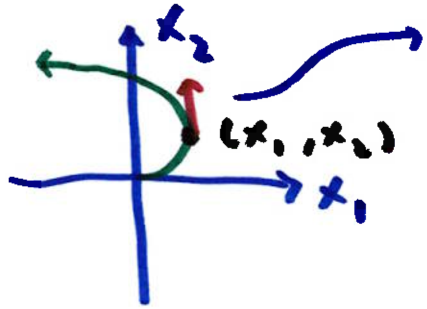
often we want to know $x_1(t)$ vs $x_2(t)$



this is called

a phase diagram

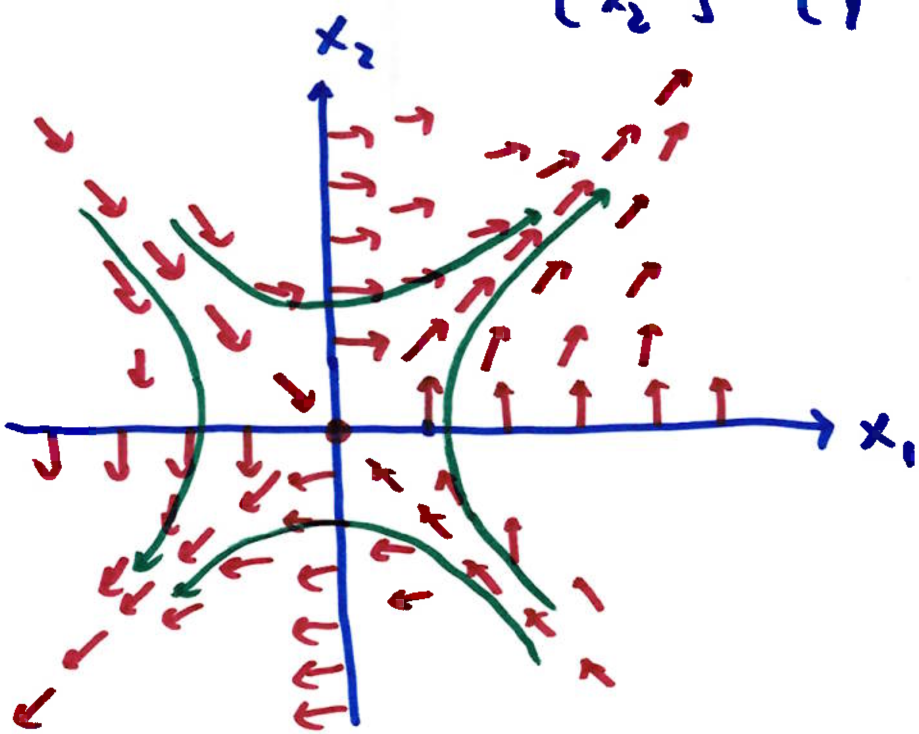
the matrix A can tell us the vectors tangent to a solution curve



$\vec{x}'$ → comes from $\vec{x}' = A\vec{x}$

for example, $\vec{x}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x}$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



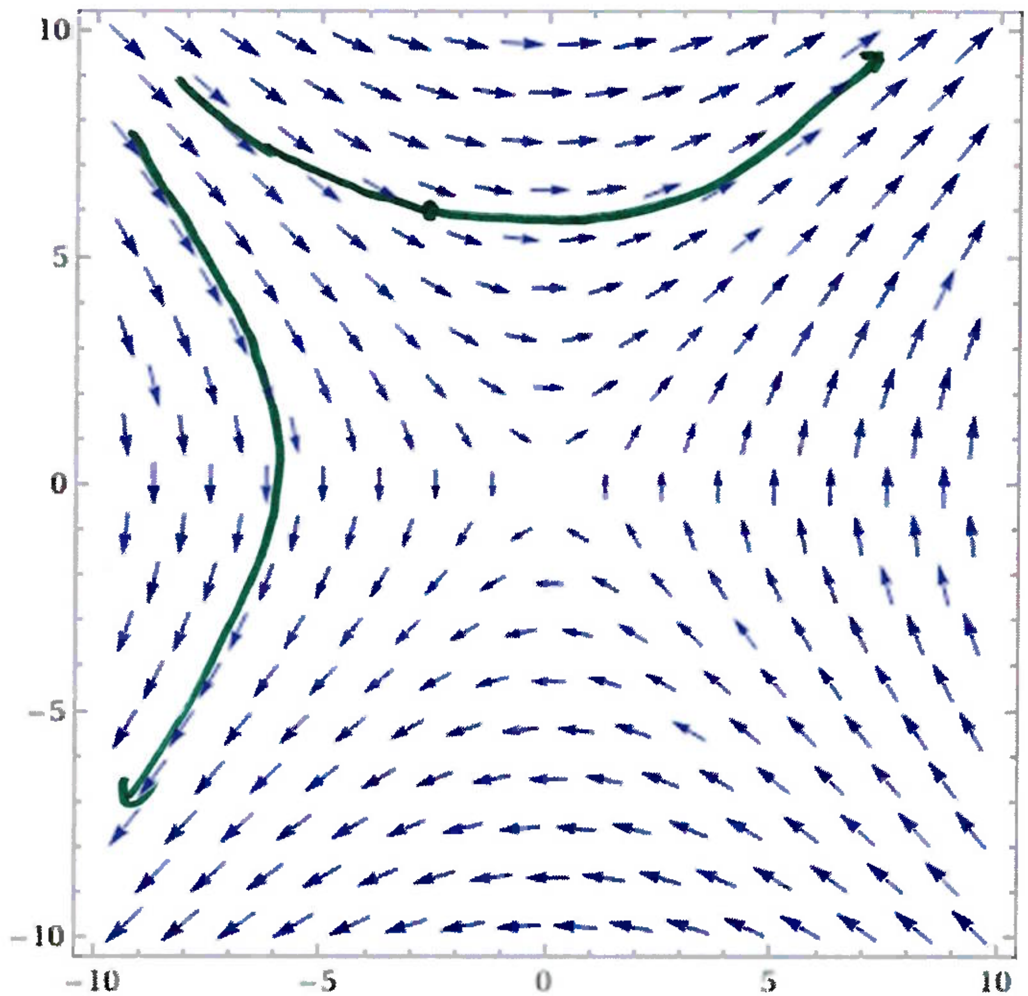
if $\vec{x} = \vec{0} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\vec{x}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

at $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{x}' = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

at $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{x}' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

at $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ $\vec{x}' = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$



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now let's find the solution quantitatively

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x} \quad \longleftrightarrow \quad \begin{matrix} x_1' = x_2 \\ x_2' = x_1 \end{matrix} \quad \longleftrightarrow \quad x_1'' - x_1 = 0$$

$$\downarrow$$

$$x_1(t) = c_1 e^{t} + c_2 e^{-t}$$
$$x_2(t) = c_1 e^{t} - c_2 e^{-t}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} : \quad \lambda_1 = 1, \ \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\lambda_2 = -1, \ \vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

so, solution is $\boxed{\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2}$

$\lambda_1, \vec{v}_1$ and $\lambda_2, \vec{v}_2$ are the eigenvalue/eigenvector pairs

<u>example</u>

$$x_1' = x_1 + 2x_2$$
$$x_2' = 2x_1 + x_2$$

$$\vec{x}' = A\vec{x} \qquad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 - 4 = 0$$
$$(1-\lambda)^2 = 4$$
$$1-\lambda = \pm 2$$
$$\lambda = -1, \quad \lambda = 3$$

<u>$\lambda = -1$</u> solve $(A-\lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\underline{\lambda = 3}$

$$\begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

general solution: $\boxed{\vec{x} = c_1 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

the two linearly indp. fundamental solutions

$$\boxed{e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ and } e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

<u>example</u>

$$x_1' = 4x_1 + 3x_2 + 3x_3$$
$$x_2' = -8x_1 - 7x_2 - 3x_3$$
$$x_3' = 8x_1 + 8x_2 + 4x_3$$

solutions: $e^{\lambda_1 t} \vec{v}_1$

$e^{\lambda_2 t} \vec{v}_2$

$e^{\lambda_3 t} \vec{v}_3$

$$\vec{x}' = \begin{bmatrix} 4 & 3 & 3 \\ -8 & -7 & -3 \\ 8 & 8 & 4 \end{bmatrix} \vec{x}$$

$$\vec{x}' = A\vec{x}$$

$$\begin{vmatrix} 4-\lambda & 3 & 3 \\ -8 & -7-\lambda & -3 \\ 8 & 8 & 4-\lambda \end{vmatrix} = 0$$

$$\vdots$$

$\lambda = -4,\ 4,\ 1$

$\vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

general solution:

$$\boxed{\vec{x} = c_1 e^{-4t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}$$

now let's look at complex λ's

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}$$

$$\begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 + 1 = 0$$

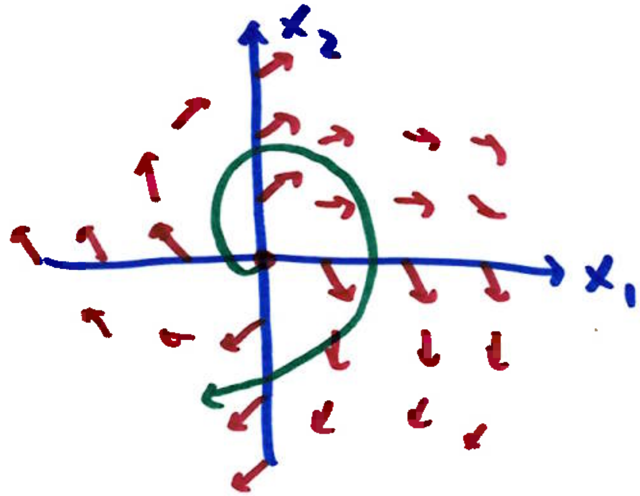
$$(1-\lambda)^2 = -1$$

$$1-\lambda = \pm i$$

$$\lambda = 1+i, \quad \lambda = 1-i$$

complex λ's come in conjugate pairs $(a+bi,\ a-bi)$

let's take a look at the phase diagram

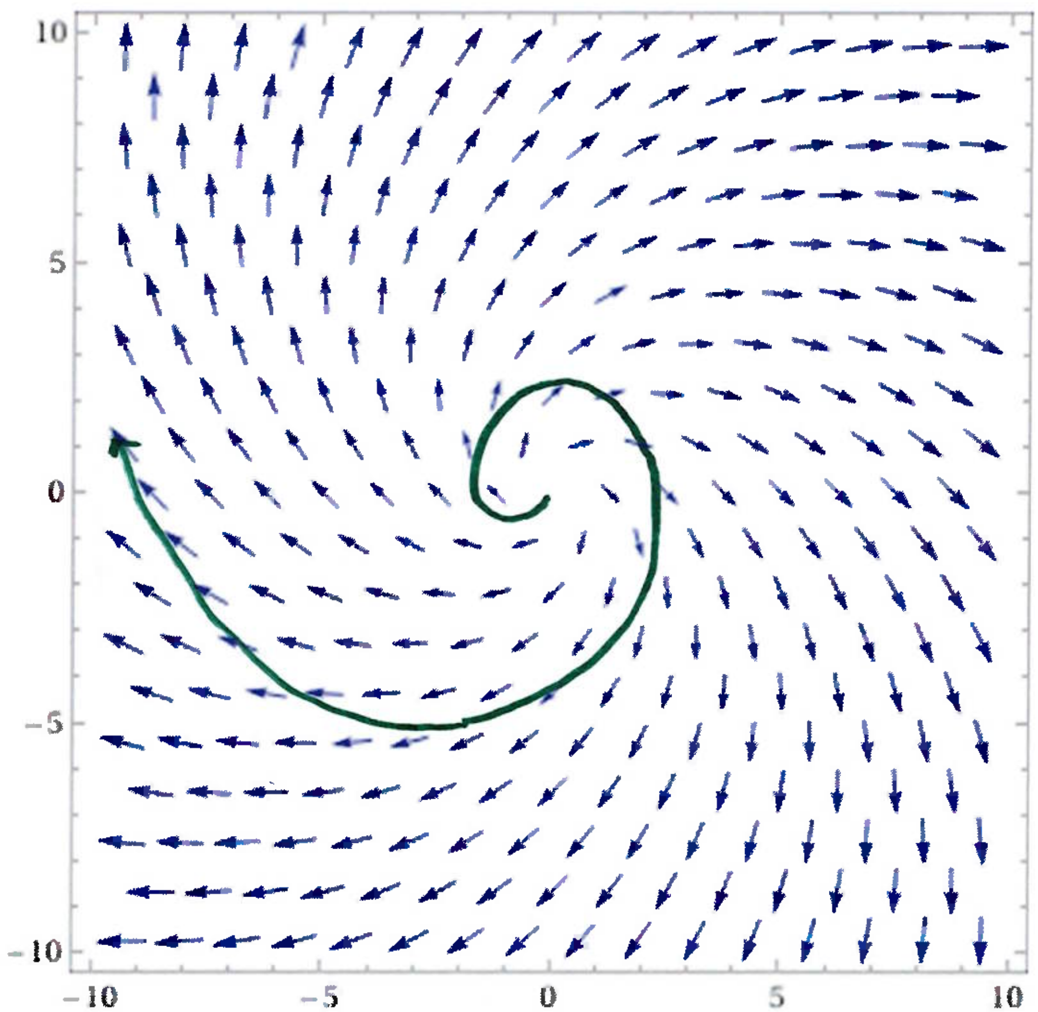


$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}$$

$\vec{x} = \vec{0} \quad \vec{x}' = \vec{0}$

$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{x}' = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

complex λ's $\rightarrow$ spirals
rotations



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$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}$$

$\lambda = 1+i,\ 1-i$

find eigenvectors

$\underline{\lambda = 1+i}$ $\quad (A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} -i & 1 & 0 \\ -1 & -i & 0 \end{bmatrix} \to \begin{bmatrix} 1 & i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

one possible eigenvector : $\vec{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$\underline{\lambda = 1-i}$ $\quad (A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} i & 1 & 0 \\ -1 & i & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

one possible eigenvector: $\vec{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

also conjugate pair

Euler's formula

$$e^{it} = \cos t + i\sin t$$

solutions: $e^{\lambda t}\vec{v}$

$\lambda_1 = 1+i$, $\vec{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ $\quad e^{\lambda_1 t}\vec{v}_1$

$$e^{(1+i)t}\begin{bmatrix} 1 \\ i \end{bmatrix} = e^t e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= e^t(\cos t + i\sin t)\begin{bmatrix} 1 \\ i \end{bmatrix}$$

generally we don't want to have "i"

$$= e^t\begin{bmatrix} \cos t + i\sin t \\ -\sin t + i\cos t \end{bmatrix} = \underbrace{e^t\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}_{\vec{u}} + i\,\underbrace{e^t\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}}_{\vec{v}}$$

$\lambda_2 = 1-i$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

⋮

same steps

⋮

$$= \underbrace{e^t\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}_{\vec{u}} - i\,\underbrace{e^t\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}}_{\vec{v}}$$

solutions are

$\vec{u} + i\vec{v}$, $\vec{u} - i\vec{v}$

we will use $\vec{u}$ and $\vec{v}$ to form the general solution

$$\vec{x} = c_1\vec{u} + c_2\vec{v}$$

general solution: $\boxed{\vec{x} = c_1 e^t \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}}$

back up a bit : $\lambda = 1+i,\ 1-i$

$\underline{\lambda = 1+i}$ $(A - \lambda I)\vec{v} = \vec{0}$

$\rightarrow \cdots \rightarrow \begin{bmatrix} 1 & i & 0 \\ 0 & 0 & 0 \end{bmatrix}$

last time we picked

$\vec{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$

let's try a different $\vec{v}$

$\vec{v} = \begin{bmatrix} i \\ -1 \end{bmatrix}$

$$e^{(1+i)t} \begin{bmatrix} i \\ -1 \end{bmatrix} = e^t e^{it} \begin{bmatrix} i \\ -1 \end{bmatrix} = e^t (\cos t + i \sin t) \begin{bmatrix} i \\ -1 \end{bmatrix}$$

$$= e^t \begin{bmatrix} -\sin t + i \cos t \\ -\cos t - i \sin t \end{bmatrix} = \underbrace{e^t \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix}}_{\vec{u}} + i \underbrace{e^t \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}_{\vec{v}}$$

general solution: $c_1\vec{u} + c_2\vec{v}$

$$\boxed{\vec{x} = c_1 e^t \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix} + c_2 e^t \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}$$

same as $\vec{x}$ on previous page (different c_1, c_2)

next time: repeated eigenvalues