7.4 Solution Curves of Linear Systems (part 1)

$$\vec{x}' = \begin{bmatrix} 4 & 7 \\ 7 & 4 \end{bmatrix} \vec{x}$$

$\lambda = -3, 11$

$\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

solution: $\vec{x}(t) = c_1 e^{-3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{11t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$x_1(t) = -c_1 e^{-3t} + c_2 e^{11t}$$

$$x_2(t) = c_1 e^{-3t} + c_2 e^{11t}$$

graph of $x_1(t)$ vs. $x_2(t)$ is called a <u>phase portrait</u>

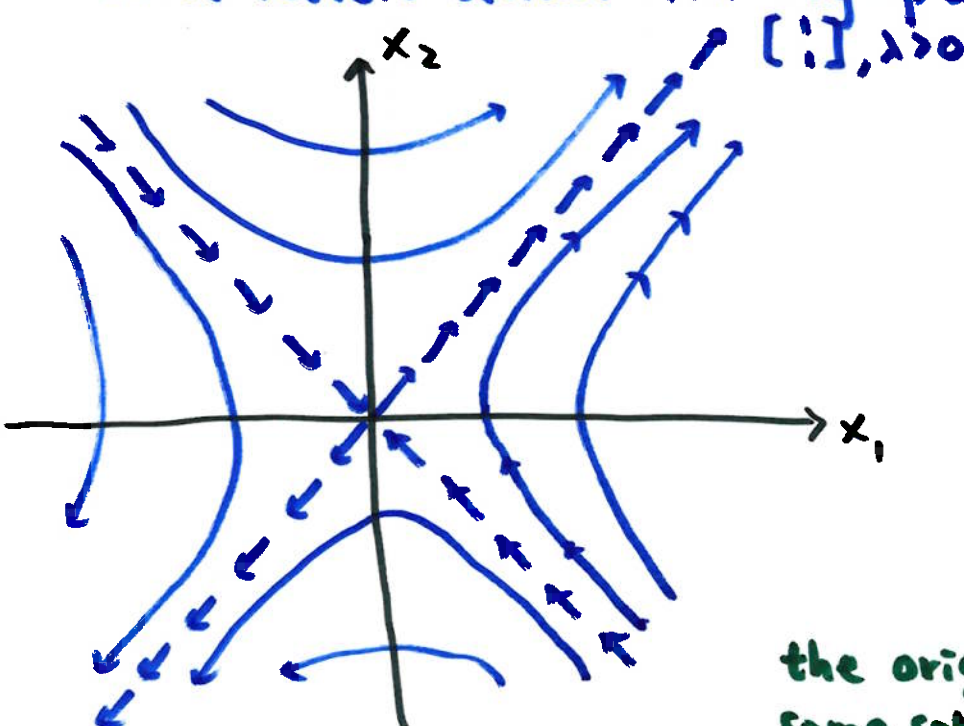
gives relationship (qualitatively) between x_1, x_2 as t changes

$$\vec{x} = \underbrace{c_1 e^{-3t}\begin{bmatrix}-1\\1\end{bmatrix}}_{\text{dominates as } t\to-\infty} + \underbrace{c_2 e^{11t}\begin{bmatrix}1\\1\end{bmatrix}}_{\text{dominate as } t\to\infty}$$

as $t\to\infty$, all solutions approach the vector $\begin{bmatrix}1\\1\end{bmatrix}$

$t\to-\infty$ " " " " $\begin{bmatrix}-1\\1\end{bmatrix}$

these vectors behave like asymptotes



if initial conditions are such that $c_1=0$, then $\vec{x} = c_2 e^{11t}\begin{bmatrix}1\\1\end{bmatrix}$

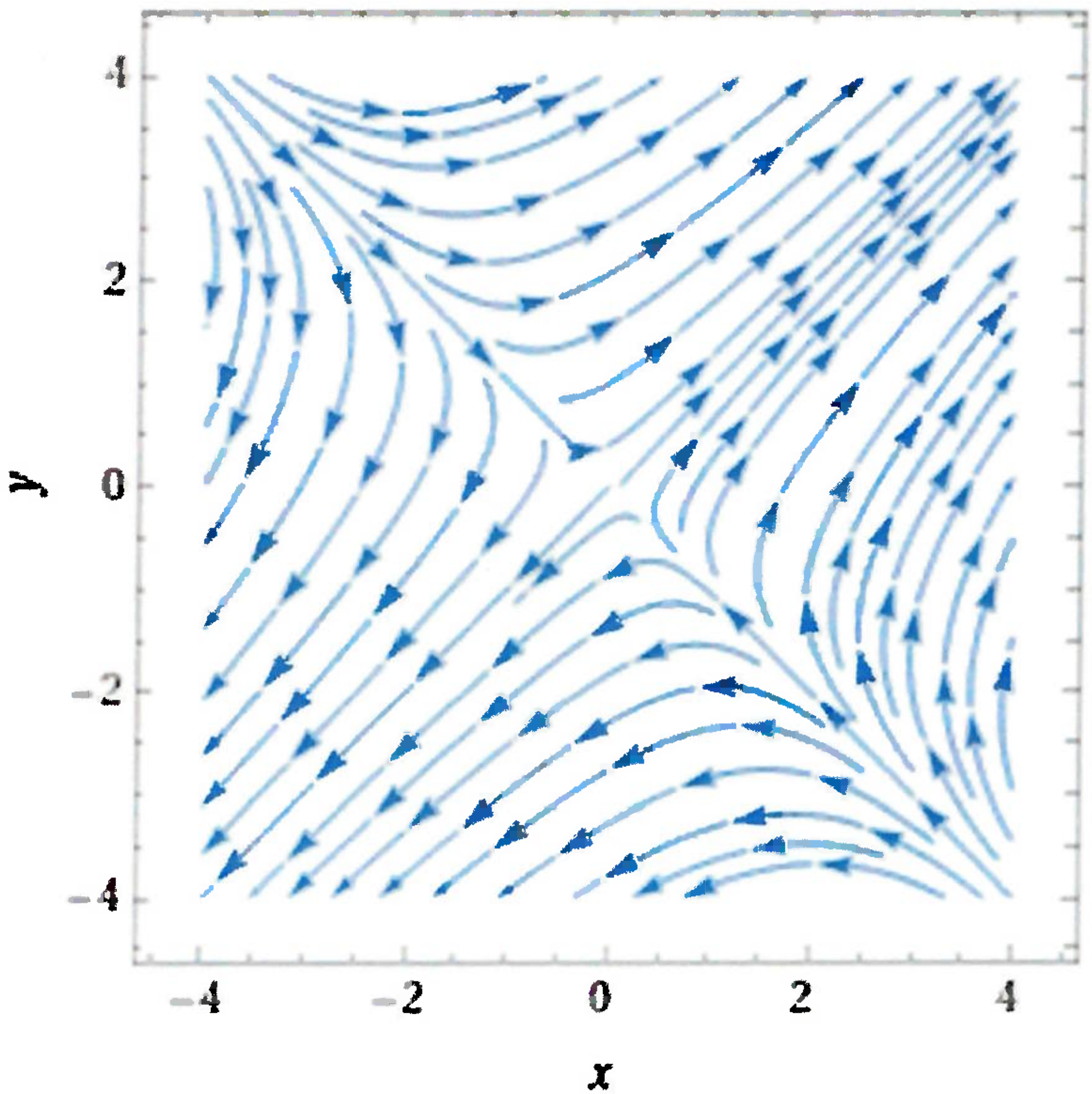
then $x_1\to\infty$, $x_2\to\infty$ as $t\to\infty$ (because $\lambda>0$) if $c_2>0$

if $c_2<0$, then to $-\infty$

for the other, following similar reasoning, solutions go toward origin because $\lambda<0$

the origin here is called a <u>saddle point</u>

some solutions go toward origin and some go away $\to$ λ's are <u>opposite</u> in signs



4
2
0
−2
−4
y
−4
−2
0
2
4
x

example

$$\vec{x}' = \begin{bmatrix} 6 & 5 \\ -3 & -2 \end{bmatrix} \vec{x}$$

$\lambda = 3, \; 1$

$\vec{v} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

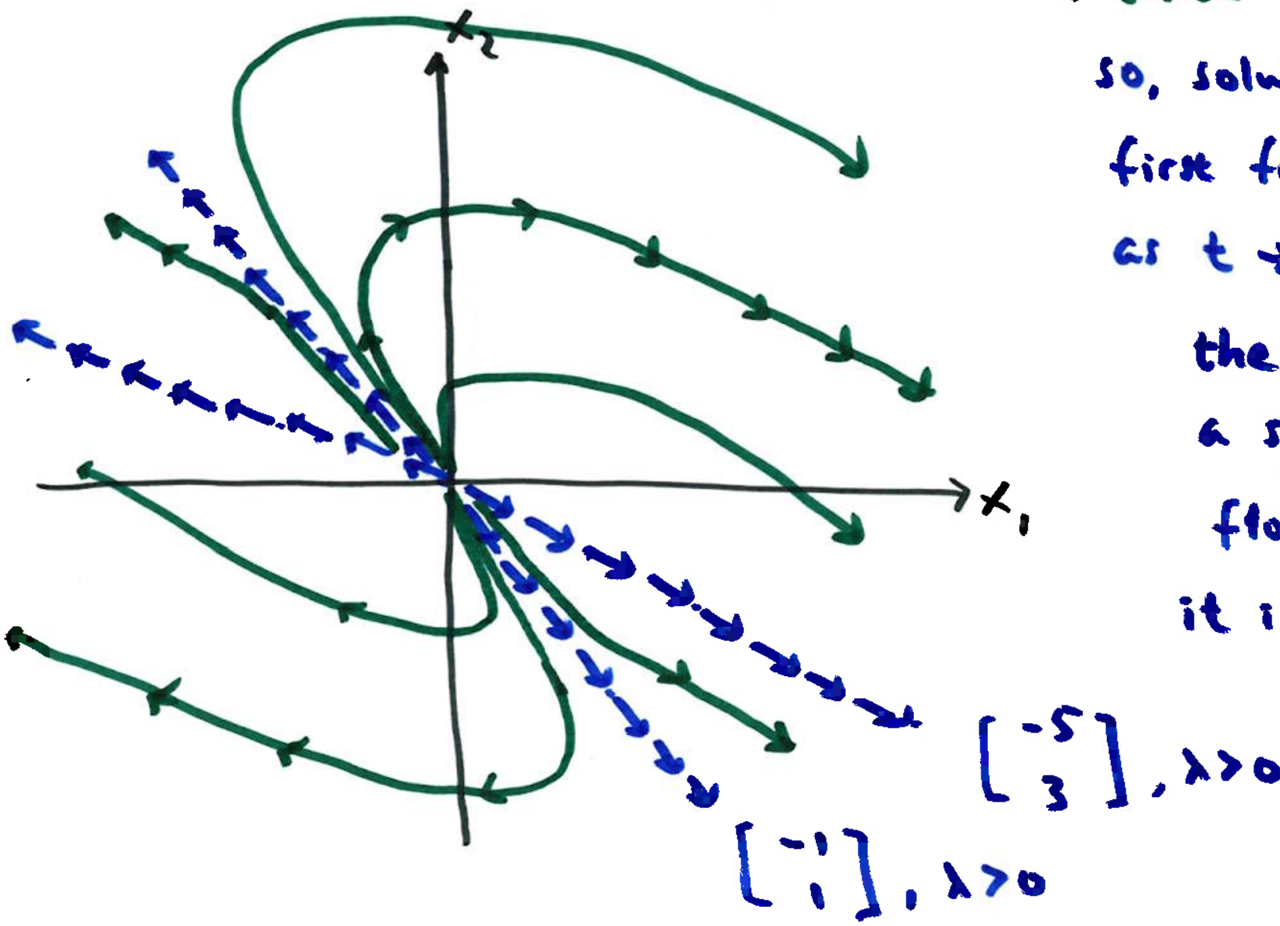
$$\vec{x} = \underbrace{c_1 e^{3t} \begin{bmatrix} -5 \\ 3 \end{bmatrix}}_{\text{dominates as } t \to \infty} + \underbrace{c_2 e^{t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\text{dominates for smaller } t}$$

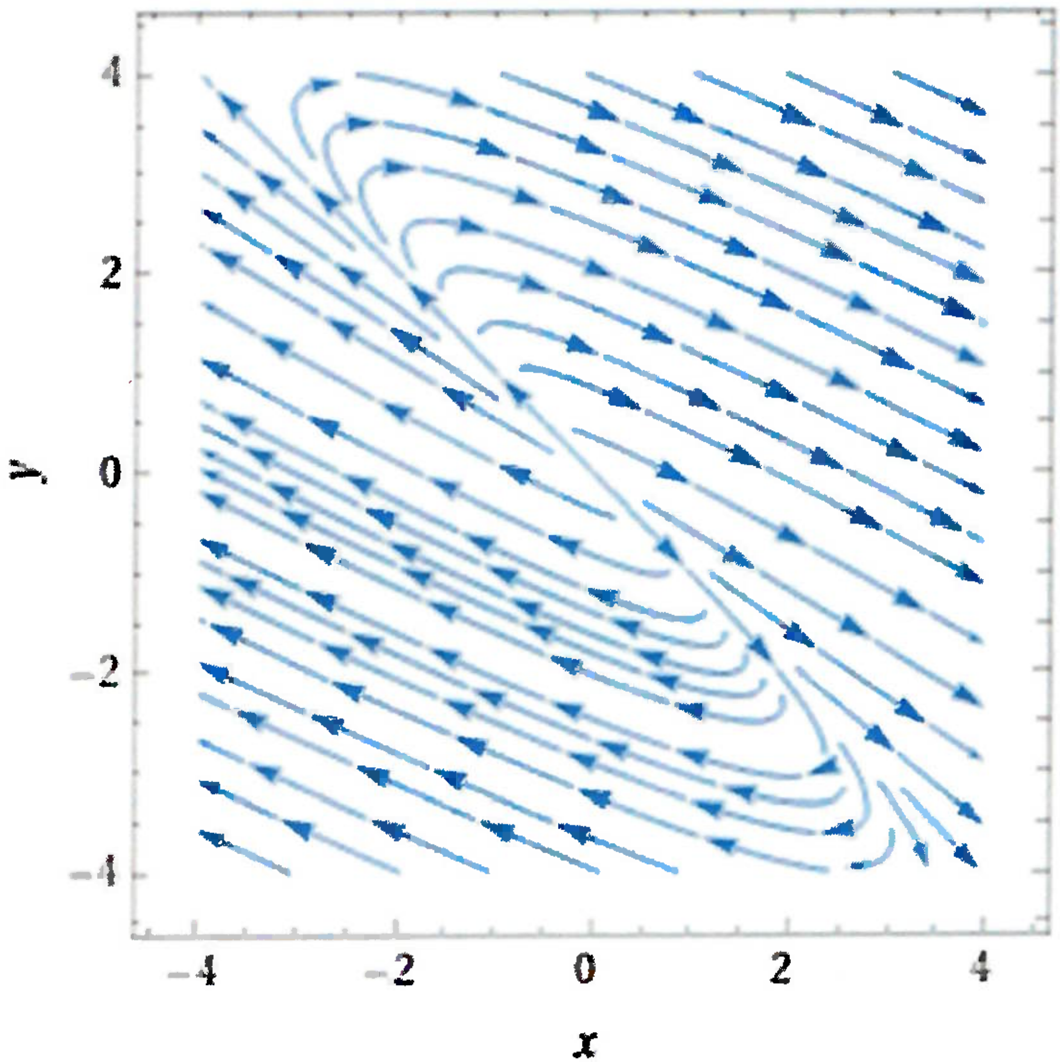
so, solutions will leave origin, first following $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, then as $t \to \infty$, following $\begin{bmatrix} -5 \\ 3 \end{bmatrix}$

the origin is called a source because things flow out of it

it is an improper nodal source

solutions all follow an asymptote near origin




4
2
0
−2
−4
y
−4
−2
0
2
4
x

if both λ's are negative, the same (or similar) picture

but all arrows go toward origin → improper nodal sink

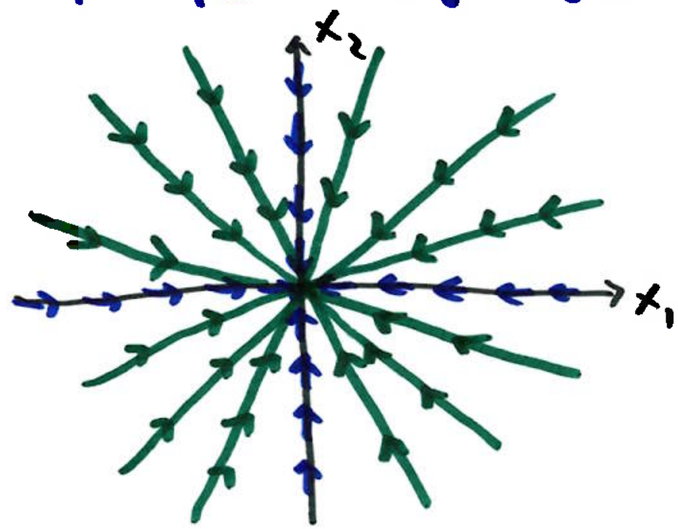
example $\quad \vec{x}' = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \vec{x} \qquad \lambda = -1, -1$

$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\vec{x} = c_1 e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$x_1 = c_1 e^{-t} \qquad x_2 = c_2 e^{-t}$

$$\frac{x_2}{x_1} = \frac{c_2}{c_1} = m \rightarrow \underbrace{x_2 = m x_1}_{\text{line thru origin w/ slope } m}$$



origin is a proper nodal sink

no asymptotes repeated λ's w/ enough eigenvectors (or matrix is complete)

$\lambda < 0$

example

$$\vec{x}' = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \vec{x}$$

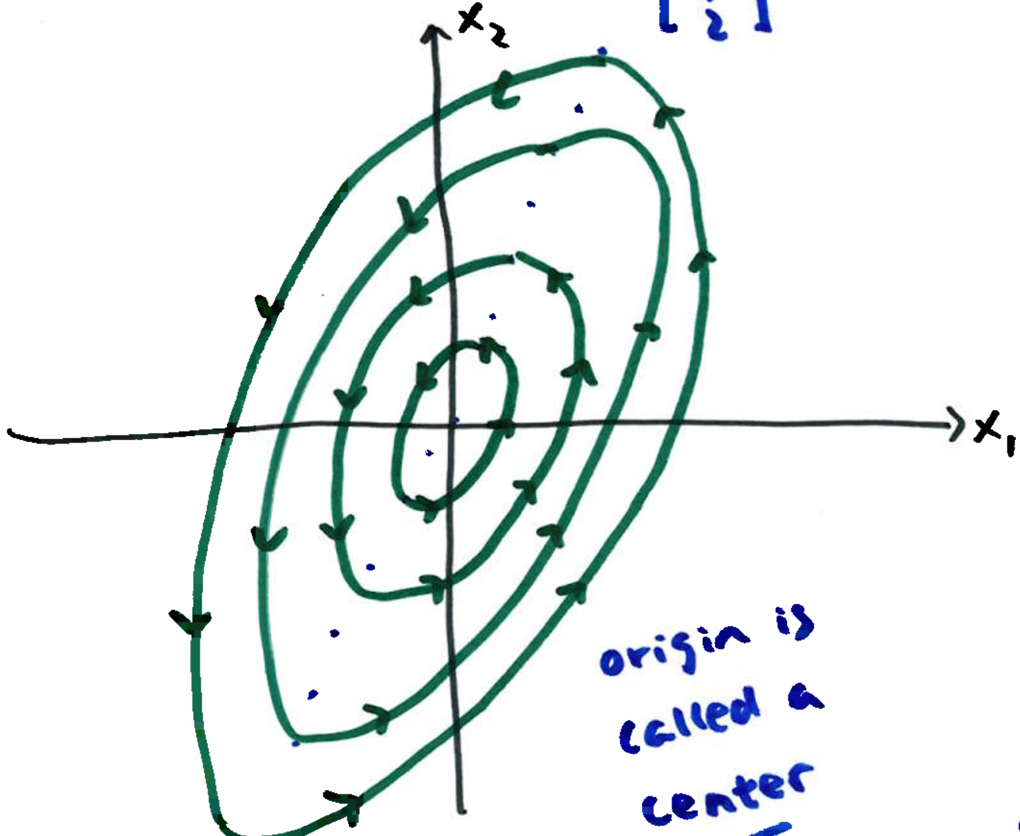
$\lambda = i, -i$

$\vec{v} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}, \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$

$$\vec{x} = c_1 \begin{bmatrix} \sin t + \cos t \\ 2\sin t \end{bmatrix} + c_2 \begin{bmatrix} -\sin t \\ \cos t - \sin t \end{bmatrix}$$

sine, cosine $\rightarrow$ periodic $\rightarrow$ rotation (solutions are ovals that go around)

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$



real part of $\vec{v}$ is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

the major axis $\leftarrow$

which direction?

ccw or cw?

pick a location, for example, $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

then $\vec{x}' = A\vec{x} = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{x}' = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ one right, two up

tangent vector

origin is called a center

example

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\vec{x}$$

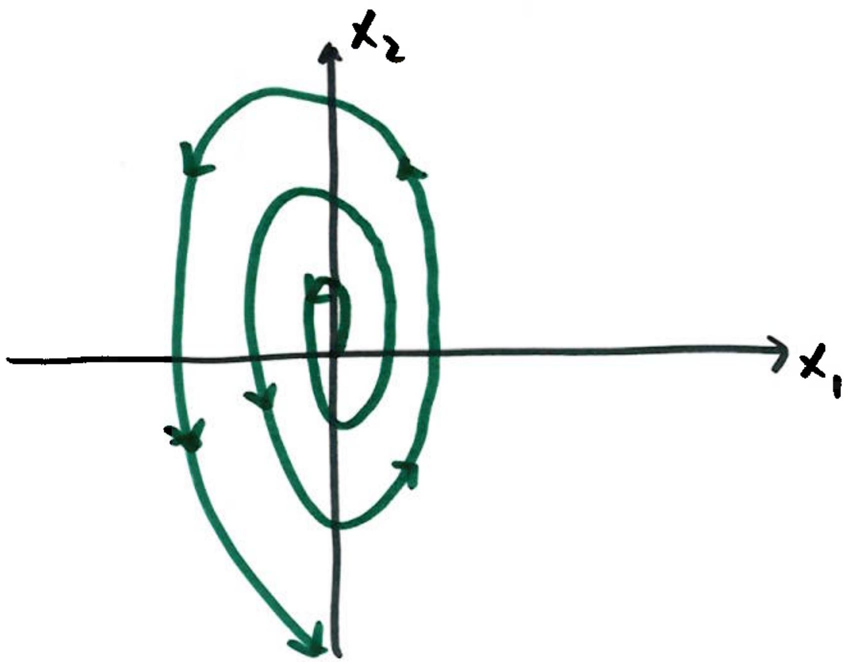
$\lambda = 1-i,\ 1+i$

$\vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}, \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$$\vec{x} = c_1 e^t \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

direction determined as in last example

due to positive real part of λ, increase as $t \to \infty$ so the curves get farther from origin as they spiral



origin here is a spiral source

(if real part of λ is <0, spiral sink