1.1 Systems of Linear Equations

why linear algebra?

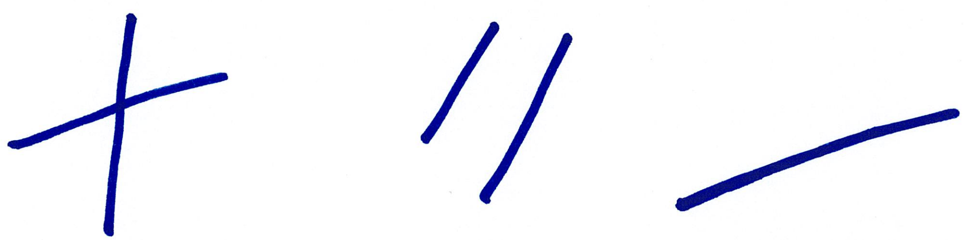
- Find the <u>solution set</u> of a system of linear eqs.
- how many solutions, if any?

examples:

$$x_1 + 2x_2 = 6$$
$$4x_1 + 7x_2 = 26$$

$(x_1, x_2) = ?$

$\Rightarrow$ intersection of two lines



one solution   none   ~~Infinif~~ infinitely-many

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 \quad - 5x_3 &= 10 \end{aligned}$$

intersection of 3 planes

back to
$$\begin{aligned} x_1 + 2x_2 &= 6 \\ 4x_1 + 7x_2 &= 26 \end{aligned}$$

in matrix notation the coefficient matrix is

$$\begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} \rightarrow \underset{\text{rows}}{2} \times \underset{\text{columns}}{2} \text{ matrix}$$

the augmented matrix of the system is

$$\begin{bmatrix} 1 & 2 & 6 \\ 4 & 7 & 26 \end{bmatrix} \rightarrow 2 \times 3 \text{ matrix}$$

solution of $x_1 + 2x_2 = 6 \rightarrow x_1 = \underbrace{6 - 2x_2}$

$4x_1 + 7x_2 = 26$

$4(6 - 2x_2) + 7x_2 = 26$

$24 - 8x_2 + 7x_2 = 26 \rightarrow x_2 = -2$

$x_1 = 6 - 2(-2) = 10$

unique solution: $(10, -2)$

matrix way: $\begin{bmatrix} 1 & 2 & 6 \\ 4 & 7 & 26 \end{bmatrix}$

eliminate x_1 (for first column) from 2nd row

multiply row 1 by -4

$$\begin{bmatrix} -4 & -8 & -24 \\ 4 & 7 & 26 \end{bmatrix}$$

add row 1 to row 2

$$\begin{bmatrix} -4 & -8 & -24 \\ 0 & -1 & +2 \end{bmatrix}$$ "triangular form"

eliminate x_2 from row 1

add to row 1 -8 times row 2

$$\begin{bmatrix} -4 & 0 & -40 \\ 0 & -1 & 2 \end{bmatrix}$$

multiply row 1 by $-1/4$

and multiply row 2 by -1

$$\begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & -2 \end{bmatrix} \end{array}$$

row 1: $1 \cdot x_1 + 0 \cdot x_2 = 10 \rightarrow x_1 = 10$

row 2: $0 \cdot x_1 + 1 \cdot x_2 = -2 \rightarrow x_2 = -2$

the things we did to the matrix are called

<u>Elementary Row Operations</u> (ERO's)

1. Add multiple of one row to another

2. Swap/interchange two rows

3. Multiply a row by any nonzero constant

ERO's do <u>NOT</u> change the solution set of the system

ERO's are reversible

two matrices are <u>row equivalent</u> if one can be transformed into another by ERO's

so $\begin{bmatrix} 1 & 2 & 6 \\ 4 & 7 & 26 \end{bmatrix}$ is row equivalent to

$$\begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & -2 \end{bmatrix}$$

careful! two **systems** are **equivalent** if they have the same solution set.

Bigger system examples

example:

$$\begin{aligned} x_2 + 4x_3 &= -3 \\ x_1 + 3x_2 + 6x_3 &= 4 \\ 2x_1 + 5x_2 + 8x_3 &= 5 \end{aligned}$$

augmented matrix:

$$\begin{bmatrix} 0 & 1 & 4 & -3 \\ 1 & 3 & 6 & 4 \\ 2 & 5 & 8 & 5 \end{bmatrix}$$

try to keep x_1 in row 1, get rid of it in other rows swap row 1 and row 2

$$\begin{bmatrix} 1 & 3 & 6 & 4 \\ 0 & 1 & 4 & -3 \\ 2 & 5 & 8 & 5 \end{bmatrix}$$

$-2 \cdot R_1 + R_3$

$$\begin{bmatrix} 1 & 3 & 6 & 4 \\ 0 & 1 & 4 & -3 \\ 0 & -1 & -4 & -3 \end{bmatrix}$$

keep x_2 in R_2, get rid of it from others

$$R_2+R_3 \quad \begin{bmatrix} 1 & 3 & 6 & 4 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

row 3: $0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = -6$

$$0 = -6$$

this means the system is <u>inconsistent</u> → <u>no solution</u>

example : the augmented matrix of a system is

$$\begin{bmatrix} 1 & -1 & 1 & 7 \\ 3 & 2 & -12 & 11 \\ 4 & 1 & -11 & 18 \end{bmatrix}$$

$-3 \cdot R_1 + R_2$ AND $-4 \cdot R_1 + R_3$

$$\begin{bmatrix} 1 & -1 & 1 & 7 \\ 0 & 5 & -15 & -10 \\ 0 & 5 & -15 & -10 \end{bmatrix}$$

$-1 \cdot R_2 + R_3$

$$\begin{bmatrix} 1 & -1 & 1 & 7 \\ 0 & 5 & -15 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

zero row $\rightarrow$ arbitrary solution in one or more variables

row 3 : $0 = 0$

$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0$ $\rightarrow$ at least one of them is arbitrary