3.3 Cramer's Rule, Volume, and Linear Transformations

another way to solve $A\vec{x} = \vec{b}$

$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 6 \\ 8 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

identity : $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

replace first column with $\vec{x}$: $I_1(\vec{x}) = \begin{bmatrix} x_1 & 0 \\ x_2 & 1 \end{bmatrix}$

multiply A by $I_1(\vec{x})$: $\begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} x_1 & 0 \\ x_2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}$

because $A\vec{x} = \vec{b}$

take determinant

$$\det\left(\begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} x_1 & 0 \\ x_2 & 1 \end{bmatrix} \right) = \det\left(\begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix} \right)$$

$$\boxed{\det\left(\begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix} \right) \det\left(\begin{bmatrix} x_1 & 0 \\ x_2 & 1 \end{bmatrix} \right) = \det\left(\begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix} \right)}$$

$$(2)(x_1) = 40 \quad \rightarrow \quad x_1 = 20$$

repeat w/ replacing 2nd col of I to eventually find $x_2 = 27$

in general, the equation in green box is

$$\det(A) \cdot x_i = \det(\underbrace{A_i(\vec{b})}_{\text{A with } i^{th} \text{ column replaced with } \vec{b}})$$

$$\boxed{x_i = \frac{\det(A_i(\vec{b}))}{\det(A)}}$$

Cramer's Rule
(unique solution only)

<u>example</u>

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ -2 & -4 & -3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -2 \\ 2 \\ 9 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

find x_2

$$x_2 = \frac{\det\left(\begin{bmatrix} 1 & -2 & 4 \\ 0 & 2 & 5 \\ -2 & 9 & -3 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ -2 & -4 & -3 \end{bmatrix}\right)} = \frac{-15}{5} = -3$$

Cramer's rule is also used to find A^{-1}

if $B = A^{-1}$

then $AB = I$

$$A\begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n\end{bmatrix} = \begin{bmatrix}\overset{\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}}{\vec{e}_1} & \underset{\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}}{\vec{e}_2} & \cdots & \vec{e}_n\end{bmatrix}$$

then solve $A\vec{b}_i = \vec{e}_i$ by Cramer's rule

in the end, it turns out

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

the C_{ji} are <u>cofactors</u> of A

→ <u>signed</u> determinants of submatrix formed by covering up j^{th} row and i^{th} column of A.

the matrix of cofactors is called the adjugate (or classical adjoint) of A $\quad \mathrm{adj}(A)$

example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \\ 3 & 4 & 5 \end{bmatrix}$$

$$\begin{vmatrix} -2 & -1 \\ 4 & 5 \end{vmatrix} = -6$$

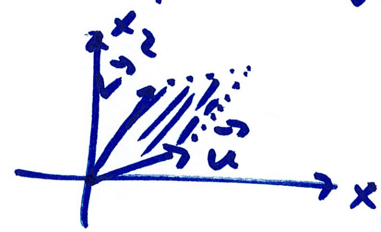
$$-\begin{vmatrix} -3 & -1 \\ 3 & 5 \end{vmatrix} = 12$$

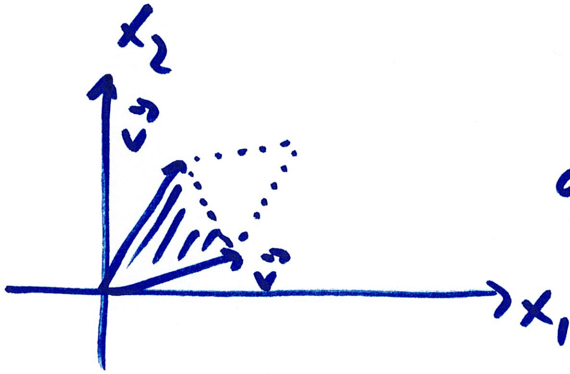
$$\operatorname{adj}(A) = \begin{bmatrix} -6 & 12 & -6 \\ 12 & -4 & 2 \\ 4 & -8 & 4 \end{bmatrix}$$

$\det(A) = 0$ $\quad A^{-1}$ does not exist

We saw last time $\left|\det[\vec{u} \;\; \vec{v}]\right|$ is area of parallelogram

$\vec{u}, \vec{v}$ in $\mathbb{R}^2$



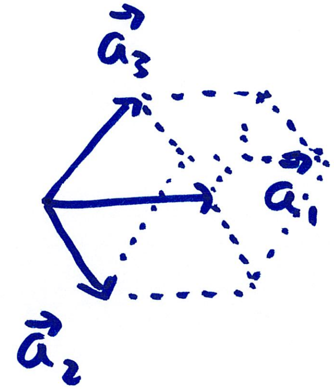


area of triangle w/ $\vec{u}, \vec{v}$ sides

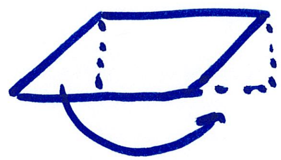
is $\dfrac{\left|\det[\vec{u} \;\; \vec{v}]\right|}{2}$

for 3x3, we get the volume of the parallelopiped

$$\left|\,\begin{vmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{vmatrix}\,\right|$$



why? Similar to



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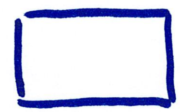
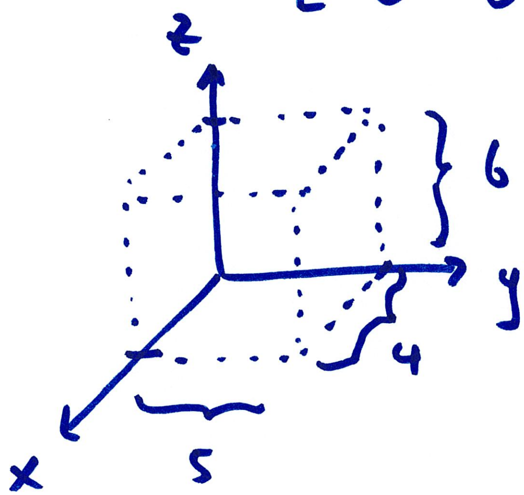
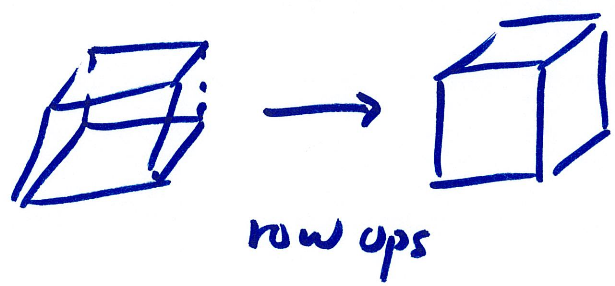


same area

consider $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ det = 120



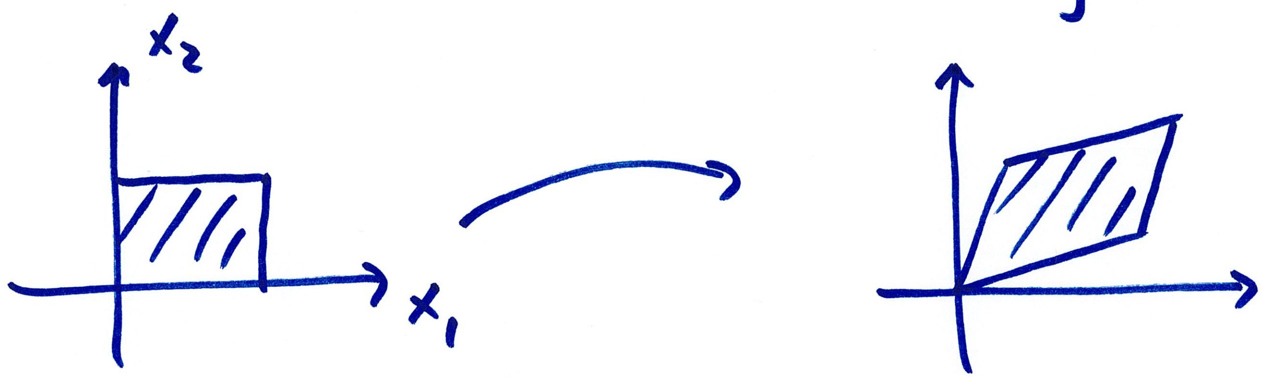
any row-equivalent matrix has the same absolute value of determinant



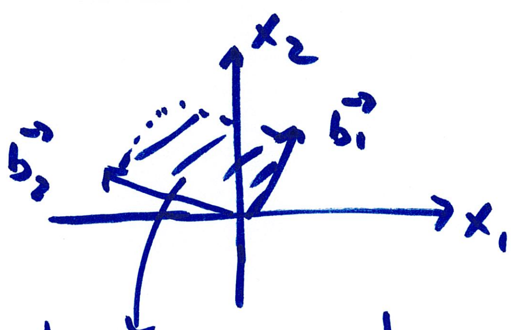
If $|\det(B)|$ is area or volume where

$$B = [\,\vec{b_1} \;\; \vec{b_2} \;\; \cdots\,]$$

then AB can be interpreted as a linear transformation of the area/volume enclosed by ~~volu~~ columns of B.



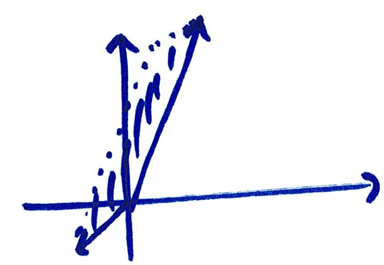
<u>example</u> $\vec{b_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{b_2} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$



$$\left| \begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix} \right| = 7$$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 \\ 11 & -5 \end{bmatrix}$$



$$||AB|| = 14 = \underbrace{\det(A)}_{\text{det of transformation matrix}} \; \underbrace{\det(B)}_{\text{area in original form}}$$

So, in general, if $T(S)$ ~~ws~~ $= AB$ is transformation of area/volume, then ~~de~~ area/volume after transformation is $\big||A||B|\big| = \big||A|\big| \cdot \{\text{area/vol of } S\}$

if $\det(A) = 0$, $\dim \operatorname{Nul} A > 0$, so at least one axes is nullified by transformation

area = 0