4.1 Vector Spaces and Subspaces

we all know what vectors in $\mathbb{R}^n$ are

other things that behave according to a set of rules can also be called "vectors" - they are in vector space

Vector Space

nonempty set of objects called "vectors" on which operations called "addition" and "multiplication by scalars" subject to the following 10 axioms. $\vec{u}, \vec{v}, \vec{w}$ are vectors and c, d are scalars.

1. $\vec{u} + \vec{v}$ is in the set
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
3. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
4. $\vec{0}$ defined such that $\vec{0} + \vec{u} = \vec{u}$
5. $-\vec{u}$ defined such that $\vec{u} + (-\vec{u}) = \vec{0}$

6. $c\vec{u}$ is in the set

7. $c(\vec{u}+\vec{v}) = c\vec{u} + c\vec{v}$

8. $(c+d)\vec{u} = c\vec{u} + d\vec{u}$

9. $c(d\vec{u}) = (cd)\vec{u}$

10. $1\vec{u} = \vec{u}$

some examples: the real number system $\mathbb{R}$
(but the set of natural numbers is not)
$M_{2\times2}$, all 2×2 matrices are in a vector space
set of all 2nd degree polynomials

$$\vec{p}(t) = a_0 + a_1 t + a_2 t^2$$

If a vector space is a subset of another vector space, then it's called a <u>subspace</u>. Only 3 of the 10 axioms need to be check.

a) existence of $\vec{0}$ (#4)
b) $\vec{u}+\vec{v}$ is in set (#1)
c). $c\vec{u}$ is in set (#6)

the rest are satisfied automatically

a subspace $\underline{\underline{is}}$ a vector space

a vector space $\underline{\underline{is}}$ a subspace

objects in subspace must be similar to the vector space ~~it~~ the subspace belongs to

for example, $\mathbb{R}^2$ is a subset of $\mathbb{R}^3$ but $\underline{\underline{NOT}}$ a subspace of $\mathbb{R}^3$

$$\mathbb{R}^2 : \begin{bmatrix} a \\ b \end{bmatrix} \qquad \mathbb{R}^3 : \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

vectors of the form $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ are in ~~a~~ subspace of $\mathbb{R}^3$

$\begin{bmatrix} a \\ b \end{bmatrix}$ does behave just like $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$

<u>example</u> Is the set of all polynomials of the form $\vec{p}(t) = at^4$ a subspace of $\mathbb{P}_4$ (all fourth-deg polynomials)?

a) Is $\vec{0}$ defined?

yes, $\vec{0} = 0t^4$

b) closed under addition?

$\vec{p}(t) = at^4 \qquad \vec{g}(t) = bt^4$

$\vec{p}(t) + \vec{g}(t) = (a+b)t^4 = ct^4$ still in the form of at^4

c). closed under scalar multiplication?

$\vec{p}(t) = at^4$

$c\vec{p}(t) = cat^4 = (ca)t^4 = dt^4$

$\vec{p}(t) = at^4$ are in a subspace

<u>example</u> All polynomials in the form $\vec{p}(t) = a + t^2$

Subspace of $\mathbb{P}_2$?

a) $\vec{0}$ defined?

$\vec{0} = 0 + 0t^2$ → is not in the form of $a + t^2$ (lost the t^2)

So these do not live in subspace of $\mathbb{P}_2$

$$\boxed{\begin{array}{l}\text{If } \vec{v}_1, \vec{v}_2, \dots, \vec{v}_p \text{ are in a vector space } V \\ \quad \text{then span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\} \text{ is a subspace of } V\end{array}}$$

$\{\vec{v}_2, \vec{v}_2, \dots, \vec{v}_p\}$ is called the <u>spanning set</u>

why? Suppose $\vec{v}_1, \vec{v}_2$ are in V. Let $H = \text{span}\{\vec{v}_1, \vec{v}_2\}$

$\vec{0} = 0\vec{v}_1 + 0\vec{v}_2$ is in H

$\vec{u} = a\vec{v}_1 + b\vec{v}_2 \qquad \vec{w} = c\vec{v}_1 + d\vec{v}_2$

$\vec{u} + \vec{w} = (a+c)\vec{v}_1 + (b+d)\vec{v}_2$ is just another linear combo of $\vec{v}_1, \vec{v}_2$

$c\vec{u} = (ca)\vec{v}_1 + (cb)\vec{v}_2$

H $\underline{\underline{\text{is}}}$ a subspace

<u>example</u> W is set of all vectors of the form

$$\begin{bmatrix} 4b-2c \\ -b \\ 9c \end{bmatrix} = b\begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -2 \\ 0 \\ 9 \end{bmatrix}$$

W is $\text{span}\left\{\begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 9 \end{bmatrix}\right\}$ and using the result from above, W is a subspace (of $\mathbb{R}^3$)