4.6 Rank

If A is $m \times n$, $\dim \operatorname{Col} A = \operatorname{rank} A = \#$ of pivot columns $= \#$ basic ~~variables~~ Variables

$\dim \operatorname{Nul} A = \#$ free variables

$\dim \operatorname{Col} A + \dim \operatorname{Nul} A = n$ (The Rank Theorem)

what about the row space of A (Row A)?

$$A = \begin{bmatrix} 1 & -4 & 1 & -7 \\ -1 & 2 & 2 & 1 \\ 2 & 4 & -16 & 22 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & -5 & 5 \\ 0 & -2 & 3 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$A \sim B$

row space of A: subspace spanned by rows of A

A and B above have the <u>same</u> row space.

We obtained B by row reductions, so rows of B are linear combos of rows of A, which means row space of B is contained in row space of A. But row operations are reversible, so we could have gotten A from B by doing ERO's.

Rows of A are therefore linear combos of rows of B. So row space of A is contained in row space of B. But if two spaces are contained within each other, they must be the same.

$$\boxed{\text{If } A \sim B, \text{ then } \operatorname{Row} A = \operatorname{Row} B}$$

The basis vectors of row space of either are the <u>non zero rows</u> of the <u>echelon matrix</u>. Not from the original matrix because row operations can change linear dependence.

here, the basis of Row A or Row B is $\{(1,0,-5,5), (0,-2,3,-6)\}$

If we want to know which rows of A are basis vectors of Row A, we can find the basis of Col A^T.

here, the first two rows of B are linearly independent, but this does <u>NOT</u> mean the first two rows of A are linearly independent.

the above implies that rank A = dim Col A = dim Row A

= dim Col A^T

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 9 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$A \sim B$

Is Col $A = \mathbb{R}^3$? Yes, 3 pivots, 3 linearly independent columns of 3 elements each, so they <u>must</u> form basis of $\mathbb{R}^3$. basis for Col $A = \left\{ \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -5 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \right\}$

Does $A\vec{x} = \vec{b}$ always have a solution for some $\vec{b}$ in $\mathbb{R}^3$?

Yes, because Col $A = \mathbb{R}^3$ so any $\vec{b}$ in $\mathbb{R}^3$ <u>is</u> a linear combo of columns of A.

basis for Row A = $\{(1,0,9,0), (0,1,-5,0), (0,0,0,1)\}$

(same dimension as Col A)

If A is 4×5 and $\underbrace{\dim \operatorname{Nul} A = 2}$. Is $\operatorname{Col} A = \mathbb{R}^3$?

$$\begin{array}{ccccc} \left[\begin{matrix} \boxed{\cdot} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \boxed{\cdot} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \boxed{\cdot} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}\right] \\ x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \end{array}$$

2 free variables

3 pivot columns, $\operatorname{rank} A = 3$

no, $\operatorname{Col} A \neq \mathbb{R}^3$ because $\mathbb{R}^3$ vectors look like $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Is $A\vec{x} = \vec{b}$ always consistent?

no, because we need 4 basis vectors/columns to have $\operatorname{Col} A = \mathbb{R}^4$ (which guarantees $A\vec{x} = \vec{b}$ is always consistent)

but we only have 3.

Can a 6×9 matrix have a $\underbrace{\text{two-dimensional null space}}$?

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

2 free variables

7 basic "

but only 6 rows here, can't place all 7

What's the relationship between Row A, Col A, Nul A, and Nul A^T?

$$A = \begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

basis for Nul A: x_3 free $\quad x_2 = -3x_3, \quad x_1 = -2x_3$

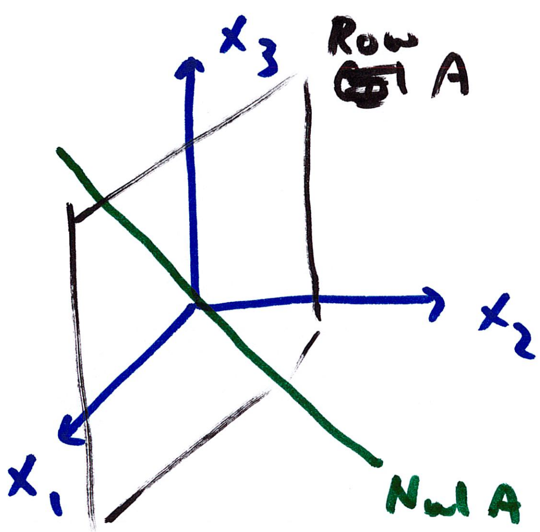
$$\text{Nul } A = \left\{ x_3 \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} \right\} \qquad \text{Nul } A = \text{span}\left\{ \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} \right\}$$

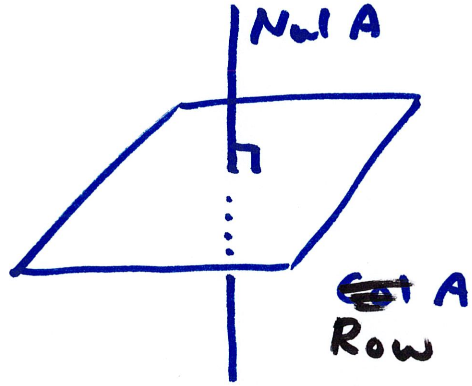
$$\text{Col } A = \text{span}\left\{ \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\text{Row } A = \text{span}\left\{ (1,0,2), (0,1,3) \right\}$$

$$A^T = \begin{bmatrix} 3 & 6 & 2 \\ -1 & 0 & 1 \\ 3 & 12 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 5/6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Nul } A^T = \text{span}\left\{ \begin{bmatrix} 1 \\ -5/6 \\ 1 \end{bmatrix} \right\}$$





Nul A $\perp$ ~~Col~~ Row A