Appendix B and 5.5 Complex Eigenvalues

HW 26 + HW 27 due together

↖ handwritten only, see course page
(do # 1, 2, 6, 8)

Imaginary part of z
$\text{Im}(z)$

A complex number is written as $z = a + bi$ a, b are real

$\boxed{i^2 = -1}$

real part of z $\text{Re}(z)$

$\mathbb{R}$: set of all real numbers

$\mathbb{C}$: " " complex numbers

$\mathbb{R}^2 = \begin{bmatrix} x \\ y \end{bmatrix}$ x, y real

$\mathbb{C}^2 = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ z_1, z_2 are complex

complex conjugate: $z = a + bi$

$\bar{z} = a - bi$

$\overline{3+4i} = 3 - 4i$

$\overline{3-4i} = 3 + 4i$

$$(a+bi) \pm (c+di) = (a+c) \pm (b+d)i$$

$$(a+bi)(c+di) = ac + adi + cbi + bd\overset{-1}{i^2}$$

$$= (ac - bd) + (ad + cb)i$$

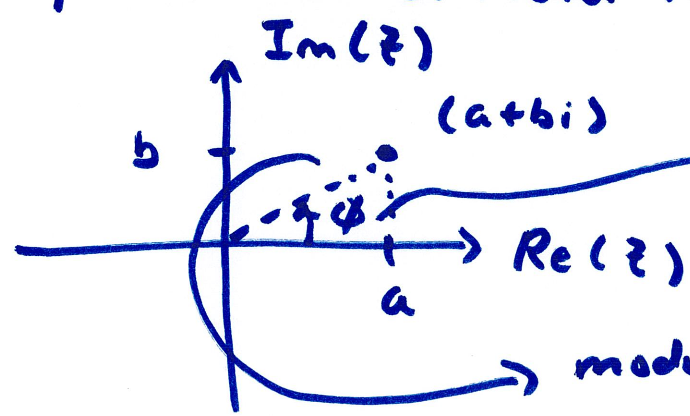
division is not like polynomials

$$\frac{1+2i}{3+4i} = \frac{1+2i}{3+4i} \cdot \frac{3-4i}{3-4i} = \frac{(1+2i)(3-4i)}{(3+4i)(3-4i)}$$

$$= \frac{3-4i+6i-8i^2}{9-12i+12i-16i^2} = \frac{11+2i}{25} = \frac{11}{25} + \frac{2}{25}i$$

$a+bi$ can be interpreted like a vector in $\mathbb{R}^2$

$z = a+bi$



$\phi = \tan^{-1}\left(\frac{b}{a}\right)$

modulus of z : $|z| = \sqrt{a^2+b^2}$

polar form : $z = a + bi = re^{i\phi}$ where r, ϕ defined as on last page

$= r(\cos\phi + i\sin\phi)$

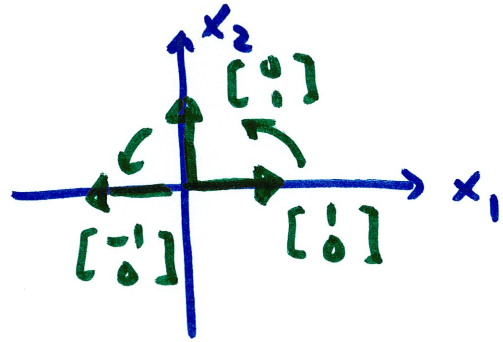
de Moivre's Theorem : $z = r(\cos\phi + i\sin\phi)$

then $z^k = r^k(\cos k\phi + i\sin k\phi)$

5.5 Complex Eigenvalues

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ what does this do to $\vec{x}$?

$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



$A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

this is a quarter circle turn counterclockwise

$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$ no vector can preserve heading

$A\vec{x} = \lambda \vec{x} \quad \lambda = ?$

find λ's: $|A - \lambda I| = 0 \qquad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \rightarrow \lambda = \pm i$$

find eigenvectors

$\lambda = i \quad (A - \lambda I)\vec{x} = \vec{0} \quad \begin{bmatrix} -i & -1 & 0 \\ 1 & -i & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \end{bmatrix}$

x_2 free $\quad x_1 = i x_2 \quad$ choose $x_2 = 1$

$$\vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$\lambda = -i \quad (A - \lambda I)\vec{x} = \vec{0} \quad \begin{bmatrix} i & -1 & 0 \\ 1 & i & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & i & 0 \\ 0 & 0 & 0 \end{bmatrix}$

x_2 free $\quad x_1 = -i x_2 \quad$ choose $x_2 = 1$

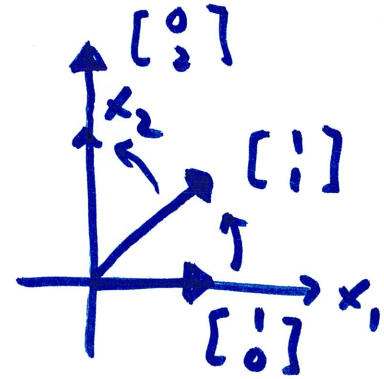
$$\vec{v} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

note λ's and $\vec{v}$'s are complex conjugate pairs.

complex λ's are associated with rotation, but there can be scaling, too.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$



$$\underbrace{\text{45° ccw turn with a lengthening by a factor of } \sqrt{2}}_{\text{eigenvalues tell us this}}$$

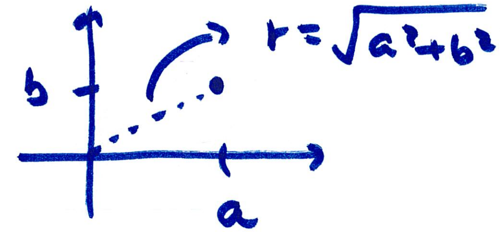
$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \qquad (1-\lambda)^2 + 1 = 0 \qquad 1-\lambda = i \quad \text{or} \quad 1-\lambda = -i$$

$$\lambda = 1-i \quad \text{or} \quad \lambda = 1+i$$

$\lambda = a \pm bi$ then the scaling factor is $\sqrt{a^2+b^2}$

$\phi = \tan^{-1}\left(\frac{b}{a}\right)$ is the turning angle rotation matrix : $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ a, b real, not both zero

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r \begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix}$$



$$= \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

$\hookrightarrow$ scaling factor

$\phi = \tan^{-1}\left(\frac{b}{a}\right)$

turning angle

<u>example</u> $A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

$\lambda = 2+i$ $\vec{v} = \begin{bmatrix} -1+i \\ 1 \end{bmatrix}$

$\lambda = 2-i$ $\vec{v} = \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$

$\lambda = a \pm bi$ here, $a=2$, $b=1$ $r = \sqrt{a^2+b^2} = \sqrt{5}$

define $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ this is hidden inside A

we can decompose A much like how we diagonalize matrices

$$A = PCP^{-1}$$

$\hookrightarrow$ not diagonal, but a rotation matrix

eigenvectors: $\begin{bmatrix} -1+i \\ 1 \end{bmatrix}, \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$ separate real and imag parts

$$\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \pm i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

use as columns of P

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} = PCP^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}}_{\text{scaling + rotation}} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A\vec{x} = \underbrace{P\underbrace{C\underbrace{P^{-1}\vec{x}}_{\text{change of coordinates/variables}}}_{\text{scale, rotate}}}_{\text{undo change of variables/ coordinates}}$$

λ, if complex, must come in pairs.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 5 & -3 \end{bmatrix} \qquad \lambda = \overbrace{-1+i, -1-i}^{\text{pairs of complex } \lambda\text{'s}}, 4$$

$$\vec{v} = \underbrace{\begin{bmatrix} 0 \\ 2/5 + 1/5\, i \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2/5 - 1/5\, i \\ 1 \end{bmatrix}}_{\text{pairs}}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$