6.2 Orthogonal Sets

a set of vectors is an <u>orthogonal set</u> if the vectors are mutually orthogonal.

$\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_p\}$ is orthogonal if $\vec{u}_i \cdot \vec{u}_j = 0$ when $i \neq j$

<u>example</u> $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \{\vec{i}, \vec{j}, \vec{k}\}$

$$\vec{i} \cdot \vec{j} = \vec{i} \cdot \vec{k} = \vec{j} \cdot \vec{k} = 0$$

<u>example</u> $\left\{ \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -17 \\ -4 \\ 1 \end{bmatrix} \right\}$ is an orthogonal set

$$\begin{bmatrix} 1 & -4 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & -4 & 1 \end{bmatrix} \begin{bmatrix} -17 \\ -4 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} -17 \\ -4 \\ 1 \end{bmatrix} = 0$$

If $S = \{\vec{u}_1, \ldots, \vec{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is <u>linearly independent</u> and a <u>basis</u> for the subspace spanned by S.

why? $S = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ ~~be~~ is an orthogonal set

(so $\vec{u}_1 \cdot \vec{u}_2 = \vec{u}_1 \cdot \vec{u}_3 = \vec{u}_2 \cdot \vec{u}_3 = 0$)

Suppose $\vec{0} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + c_3 \vec{u}_3$ for some c_1, c_2, c_3

$$\vec{0} \cdot \vec{u}_1 = c_1 \vec{u}_1 \cdot \vec{u}_1 + c_2 \cancel{\vec{u}_2 \cdot \vec{u}_1} + c_3 \cancel{\vec{u}_3 \cdot \vec{u}_1} \longrightarrow c_1 = 0$$

$$\vec{0} \cdot \vec{u}_2 = c_1 \cancel{\vec{u}_1 \cdot \vec{u}_2} + c_2 \vec{u}_2 \cdot \vec{u}_2 + c_3 \cancel{\vec{u}_3 \cdot \vec{u}_2} \longrightarrow c_2 = 0$$

$$\vec{0} \cdot \vec{u}_3 = c_1 \cancel{\vec{u}_1 \cdot \vec{u}_3} + c_2 \cancel{\vec{u}_2 \cdot \vec{u}_3} + c_3 \vec{u}_3 \cdot \vec{u}_3 \rightarrow c_3 = 0$$

so, $c_1 = c_2 = c_3 = 0$ is the only way

$\vec{0} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + c_3 \vec{u}_3 \rightarrow \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is linearly indp.

If a basis is orthogonal, then it is an <u>orthogonal basis</u>

→ makes calculating the weights of a linear combo easy.

<u>example</u> Orthogonal basis : $\left\{ \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \end{bmatrix} \right\} = \{ \vec{u}_1, \vec{u}_2 \}$

$\vec{x} = \begin{bmatrix} 9 \\ -7 \end{bmatrix}$ $\quad c_1 \begin{bmatrix} 2 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ -7 \end{bmatrix}$

$c_1 = ? \quad c_2 = ?$

"old" way: form augmented matrix, then row reduce.

another way: take advantage of orthogonality

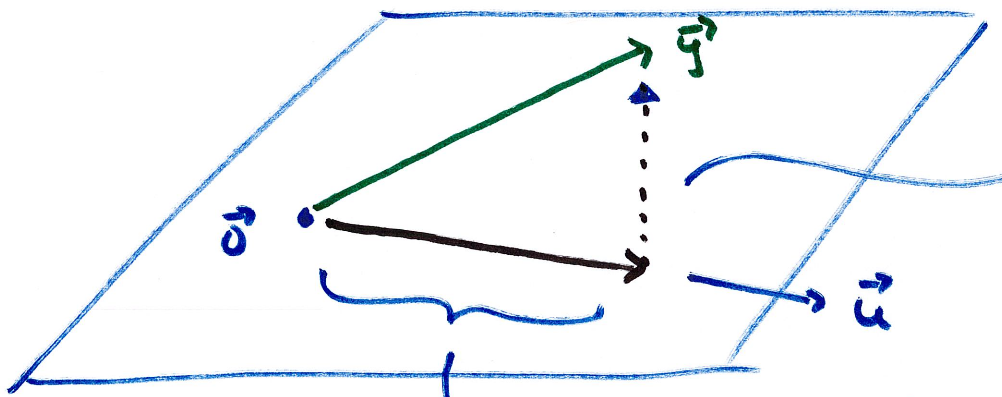
$$\vec{x} = c_1 \vec{u}_1 + c_2 \vec{u}_2$$

$$\vec{x} \cdot \vec{u}_1 = c_1 \vec{u}_1 \cdot \vec{u}_1 + \cancel{c_2 \vec{u}_2 \cdot \vec{u}_1} \rightarrow c_1 = \frac{\vec{x} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}$$

$$\vec{x} \cdot \vec{u}_2 = \cancel{c_1 \vec{u}_1 \cdot \vec{u}_2} + c_2 \vec{u}_2 \cdot \vec{u}_2 \rightarrow c_2 = \frac{\vec{x} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}$$

$$c_1 = \frac{\begin{bmatrix} 9 & -7 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix}}{\begin{bmatrix} 2 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix}} = \frac{39}{13} = 3$$

$$c_2 = \frac{\begin{bmatrix} 9 & -7 \end{bmatrix}\begin{bmatrix} 6 \\ 4 \end{bmatrix}}{\begin{bmatrix} 6 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 4 \end{bmatrix}} = \frac{26}{52} = \frac{1}{2}$$



component of $\vec{y}$ orthogonal to $\vec{u}$ $\longrightarrow$ $\vec{z}$

"Shadow" of $\vec{y}$: orthogonal projection of $\vec{y}$ onto $\vec{u}$ $\longrightarrow$ $\hat{y}$ ("y hat")

$\hat{y} + \vec{z} = \vec{y}$ let $\hat{y} = \alpha\vec{u}$

then $\vec{z} = \vec{y} - \hat{y} = \vec{y} - \alpha\vec{u}$

& $\vec{z}\cdot\vec{u} = 0 = (\vec{y} - \alpha\vec{u})\cdot\vec{u} = \vec{y}\cdot\vec{u} - \alpha\,\vec{u}\cdot\vec{u}$

$$\alpha = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}} \qquad \text{therefore,} \qquad \boxed{\hat{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\,\vec{u}}$$

orthogonal projection of $\vec{y}$ onto subspace spanned by $\vec{u}$

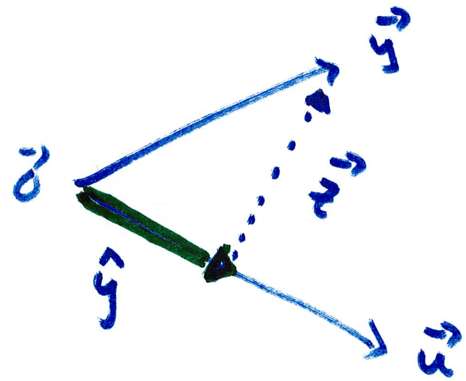
length of $\vec{u}$ doesn't matter

let $\vec{u} = c\vec{u}$

$$\hat{y} = \frac{\vec{y}\cdot c\vec{u}}{c\vec{u}\cdot c\vec{u}}\,c\vec{u} = \frac{\cancel{c}\;\vec{y}\cdot\vec{u}}{\cancel{c^2}\;\vec{u}\cdot\vec{u}}\,\cancel{c}\,\vec{u}$$

<u>example</u> $\vec{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \vec{u} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$

write $\vec{y}$ as $\cancel{z}$ $\vec{y} = \hat{y} + \vec{z}$



$$\hat{y} = \frac{\begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -5 \end{bmatrix}}{\begin{bmatrix} 1 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ -5 \end{bmatrix}}\begin{bmatrix} 1 \\ -5 \end{bmatrix} = \frac{-13}{26}\begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ 5/2 \end{bmatrix}$$

$$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 5/2 \end{bmatrix}$$

$$= \begin{bmatrix} 5/2 \\ 1/2 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \overbrace{\begin{bmatrix} -1/2 \\ 5/2 \end{bmatrix}}^{\hat{y}} + \overbrace{\begin{bmatrix} 5/2 \\ 1/2 \end{bmatrix}}^{\vec{z}}$$

notice $\|\vec{z}\|$ is the shortest distance from $(2, 3)$ to the line through $(0, 0)$ and $(1, -5)$

If $\{\vec{u}_1, \dots, \vec{u}_p\}$ is an orthogonal set and $\|\vec{u}_i\| = 1$ for all i, then the set is an <u>orthonormal set</u>

orthogonal $\rightarrow$ lin. indp.

lin. indp. $\rightarrow$ orthogonal ? No,

lin indp. but not orthogonal