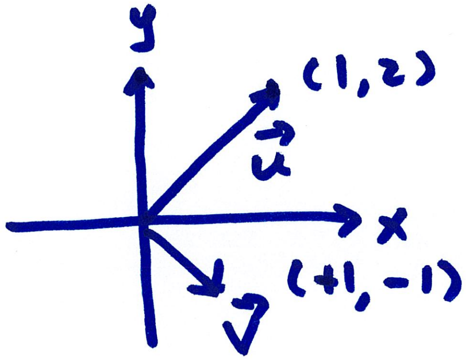
1.3 Vector Equations



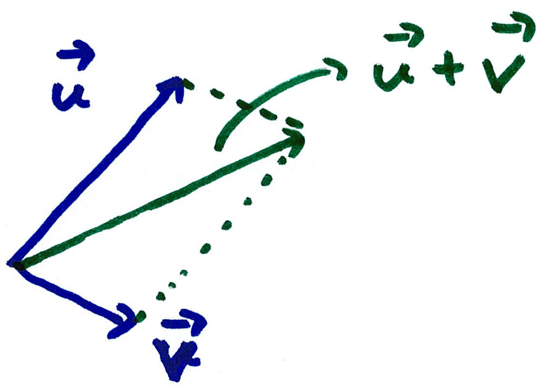
$$\vec{u} = 1\vec{i} + 2\vec{j} = \langle 1, 2 \rangle$$

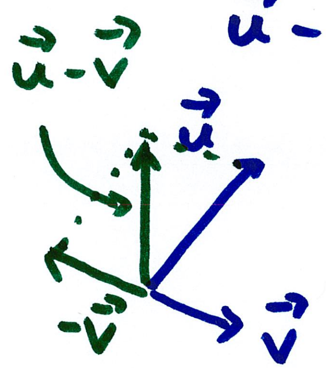
$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

matrix w/ just one column
→ column vector

$$\vec{v} = \begin{bmatrix} +1 \\ -1 \end{bmatrix}$$

$$\vec{u} + \vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$





$$\vec{u} - \vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$3\vec{u} + 2\vec{v}$$

$$= 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

any vector in the form $a\vec{u} + b\vec{v}$ is called a linear combination of $\vec{u}$ and $\vec{v}$

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}$$

linear combination of these

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ 0 \\ x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 5x_2 \\ 2x_2 \end{bmatrix} + \begin{bmatrix} -x_3 \\ 4x_3 \\ -3x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + 3x_2 - x_3 \\ 5x_2 + 4x_3 \\ x_1 + 2x_2 - 3x_3 \end{bmatrix}$$

is a vector that is linear combo of $\vec{u}, \vec{v}, \vec{w}$

looks like a system

<u>example</u>

$$\begin{aligned} 4x_1 + x_2 + 3x_3 &= 9 \\ x_1 - 7x_2 - 2x_3 &= 2 \\ 8x_1 + 6x_2 - 5x_3 &= 15 \end{aligned}$$

rewrite:

$$\begin{bmatrix} 4x_1 + x_2 + 3x_3 \\ x_1 - 7x_2 - 2x_3 \\ 8x_1 + 6x_2 - 5x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}$$

equivalent to

$$\left[\begin{array}{cccc} \boxed{\begin{matrix} 4 \\ 1 \\ 8 \end{matrix}} & \boxed{\begin{matrix} 1 \\ -7 \\ 6 \end{matrix}} & \boxed{\begin{matrix} 3 \\ -2 \\ -5 \end{matrix}} & \begin{matrix} 9 \\ 2 \\ 15 \end{matrix} \end{array}\right]$$

So, solving a system is the same as asking if the right hand side vector is a <u>linear combo</u> of the columns of the coefficient matrix __example__ Is $\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ a linear combo of

$$\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \text{ and } \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix} ?$$

can we find x_1, x_2, x_3 such that

$$\underbrace{x_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}}\ ?$$

is the same as

$$\begin{aligned} x_1 + 0x_2 + 5x_3 &= 2 \\ -2x_1 + x_2 - 6x_3 &= -1 \\ 0x_1 + 2x_2 + 8x_3 &= 6 \end{aligned}$$

and

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{bmatrix}$$

$$\sim \cdots \sim \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 is free

$x_2 + 4x_3 = 3 \rightarrow x_2 = 3 - 4x_3$

$x_1 + 5x_3 = 2 \rightarrow x_1 = 2 - 5x_3$

at least one solution exists, so

$\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ $\underline{\underline{\text{is}}}$ a linear combo of

$$\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}$$

one way: $x_3 = 1$ (arbitray)

$x_2 = -1$

$x_1 = -3$

so $-3\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$

example

is $\begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix}$ a linear combo of

$\begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} +1 \\ -3 \\ -8 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 12 \end{bmatrix}$?

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & 8 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{bmatrix}$$

no solution

no, not a linear combo.

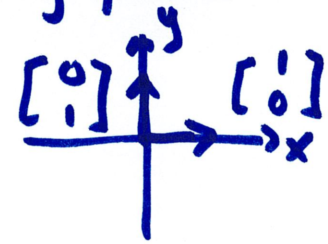
let $\vec{a_1} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ $\vec{a_2} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ $\vec{a_3} = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}$

and from earlier, ~~we~~ we know $\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ is a linear combo of $\vec{a_1}, \vec{a_2}, \vec{a_3}$

the set of all ~~vector~~ possible linear combos of $\vec{a_1}, \vec{a_2}, \vec{a_3}$ is called the subset <u>spanned</u> by $\vec{a_1}, \vec{a_2}, \vec{a_3}$, written as $\mathrm{span}\{\vec{a_1}, \vec{a_2}, \vec{a_3}\}$

so, $\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ is in ~~the~~ $\mathrm{span}\{\vec{a_1}, \vec{a_2}, \vec{a_3}\}$

examples: what is $\mathrm{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ xy plane



what is $\mathrm{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$? y-axis

what is $\mathrm{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right\}$ x-axis, same as $\mathrm{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$