2.9 Dimension and Rank (NOT ON EXAM 1)

basis : the minimum set of vectors needed to span a subspace
these vectors ("bases") can also be used as
the coordinate system in the subspace.

<u>example</u> The basis of a subspace is

$$B = \left\{ \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -7 \\ 5 \end{bmatrix} \right\}$$

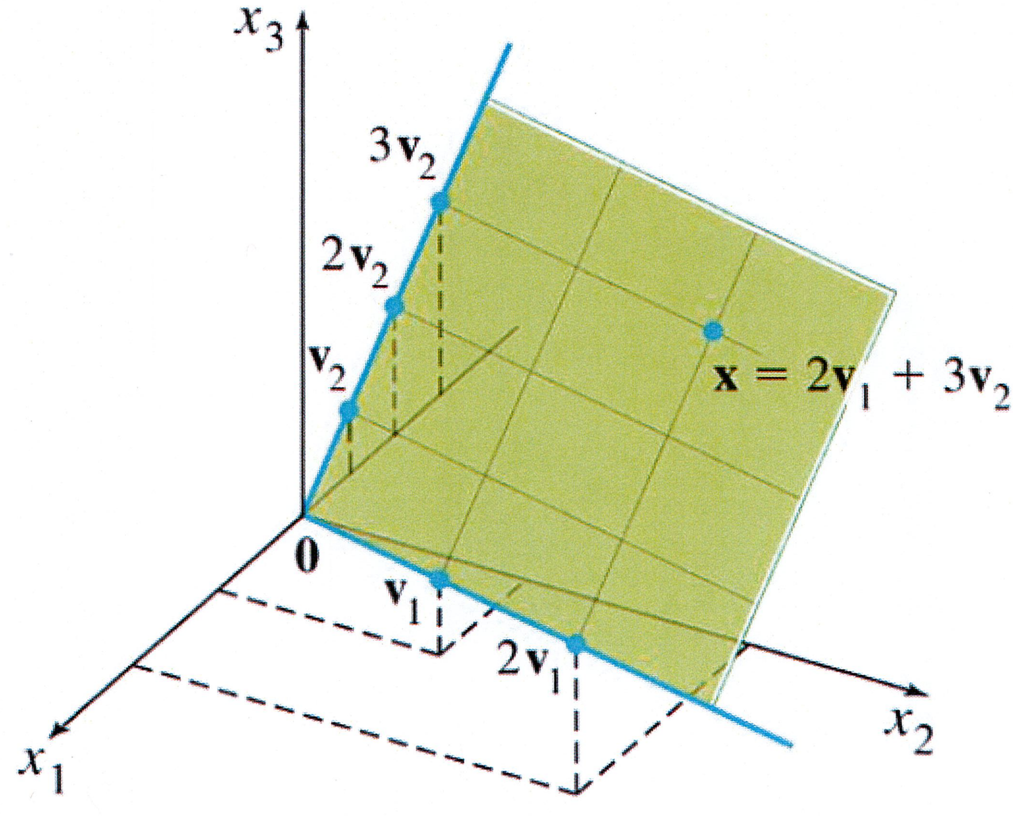
plane through origin

~~the~~ a vector in the subspace is $\begin{bmatrix} -5 \\ -17 \\ 12 \end{bmatrix}$

$$\begin{bmatrix} -5 \\ -17 \\ 12 \end{bmatrix} = (1) \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} + (3) \begin{bmatrix} -2 \\ -7 \\ 5 \end{bmatrix}$$

but it is also

$$\begin{bmatrix} -5 \\ -17 \\ 12 \end{bmatrix} = (-5) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-17) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (12) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$


x_3
$3\mathbf{v}_2$
$2\mathbf{v}_2$
$\mathbf{v}_2$
$\mathbf{x} = 2\mathbf{v}_1 + 3\mathbf{v}_2$
$\mathbf{0}$
$\mathbf{v}_1$
$2\mathbf{v}_1$
x_2
x_1

the coordinates in $\mathbb{R}^3$ is $\begin{bmatrix} -5 \\ -17 \\ 12 \end{bmatrix}$

but in B it is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\Rightarrow$ coordinates relative to the basis B or B-coordinate vector

~~not~~ really, the basis vectors can simply be looked at as a coord. transformation.

B, in this example, is a subspace of $\mathbb{R}^3$ and is a plane and it behaves just like $\mathbb{R}^2$ even though it is not $\mathbb{R}^2$.

there is a one-to-one correspondence between B and $\mathbb{R}^2$

the subspace preserves linear combination

(looks and acts like $\mathbb{R}^2$)

$\Rightarrow$ "isomorphism"

the transformation between B and $\mathbb{R}^2$ is both onto and one-to-one the basis itself is not unique

but once chosen, every vector can only be described one way

→ this is because basis vectors are linearly independent

if $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is basis set

if we could describe a vector in more than one way,

$$\vec{b} = c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}$$

$$\vec{b} = d_1\vec{v_1} + d_2\vec{v_2} + d_3\vec{v_3}$$

where $c_i \neq d_i$

then

$$\vec{0} = (c_1 - d_1)\vec{v_1} + (c_2 - d_2)\vec{v_2} + (c_3 - d_3)\vec{v_3}$$

but this cannot happen because $\vec{V_i}$ are linearly independent

thus, the $c_i \neq d_i$ assumption is wrong.

The number of vectors in a basis set of subspace H is called the <u>dimension</u> of H. $\dim H$

e.g. for $\mathbb{R}^3$, one possible basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\dim \mathbb{R}^3 = 3$

> If H is an n-dimensional subspace then <u>any</u> set of n linearly independent vectors is a basis of H.

$$\begin{bmatrix} 1 & -1 & 5 \\ 2 & 0 & 7 \\ -3 & -5 & -3 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \boxed{1} & -1 & 5 \\ 0 & \boxed{2} & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

what is $\dim \operatorname{Col} A$? 2 because columns 1 and 2 are pivot columns and are independent, so they form a basis of Col A.

basic variable

what is $\dim \underbrace{\operatorname{Nul} A}_{\text{all solutions of } A\vec{x}=\vec{0}}$? 1 because ~~so~~ there is one free variable and all solutions are multiples ~~one~~ one vector of

free variables

$\dim \text{Nul } A = \#$ of free variables

What is $\dim \text{Nul } A$ if $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$?

$\text{Nul } A = \{\vec{0}\}$ we defined $\dim \text{Nul } A = 0$

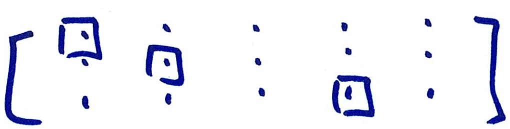
n variables

If A ~~is~~ ~~mxn~~ has n columns,

then $\underbrace{\dim \text{Col } A}_{\#\text{ of basic}} + \underbrace{\dim \text{Nul } A}_{\#\text{ of free}} = n$

- $\dim \text{Col } A$ is also called the <u>rank of A</u>

~~How~~ 3×5 matrix can have at most 3 basic variables and at least 2 free variables



at most 3 pivots

if A is 3×5 then $T : \mathbb{R}^5 \to \mathbb{R}^3$

dim Nul A tells us how many axes are "lost" due to the transformation

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$ this axis is lost (is in null space)

the $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$