MA 265 Final Exam

- Thursday, 8/2
- 3:30pm-5:30pm
- FRNY G140

- 20 multiple choice problems
- Covers all lessons
  - no special emphasis on material since exam 2

7.1 Diagonalization of Symmetric Matrices (continued)

If A is symmetric ($A^T = A$) then $A = PDP^{-1} = PDP^T$ where P is an orthogonal matrix whose columns are orthonormal eigenvectors of A and D is a diagonal matrix with the eigenvalues on the main diagonal.

<u>example</u>

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

repeated

$\lambda = 1, 4$

$1, 1, 4$

<u>$\lambda = 1$</u> $(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_2 free x_3 free $x_1 = x_2 - x_3$

$$\vec{x} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\underline{\lambda = 4}$

$$\begin{bmatrix} -2 & -1 & 1 & 0 \\ -1 & -2 & -1 & 0 \\ 1 & -1 & -2 & 0 \end{bmatrix} \sim \cdots \qquad \vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

normalize them $\vec{u}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

$$\vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$\vec{u}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

make $\vec{u}_2$ orthogonal to the rest

$$\vec{w}_2 = \vec{u}_2 - \langle \vec{u}_2, \vec{u}_1 \rangle \vec{u}_1 - \langle \vec{u}_2, \vec{u}_3 \rangle \vec{u}_3$$

$$= \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -3/2\sqrt{2} \\ -1/2\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

normalize it : $\vec{w}_2 = \cdots = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

$$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 2/\sqrt{6} \\ 0 & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Some properties of symmetric matrices

If A is symmetric, then $(A\vec{x})\cdot\vec{y} = \vec{x}\cdot(A\vec{y})$

why? If A is symmetric, then $A = A^T$

$$(A\vec{x})\cdot\vec{y} = (A\vec{x})^T\vec{y} = \vec{x}^T A^T \vec{y}$$

$$= \vec{x}^T(A\vec{y})$$

$$= \vec{x}\cdot(A\vec{y})$$

If A is symmetric, A^2 is symmetric

why? $A = A^T$

$$(A^2)^T = (AA)^T = A^T A^T = (A^T)^2 = A^2$$

what about A^3?

$$(A^3)^T = (AA^2)^T = (A^2)^T A^T = A^2 A = A^3$$

If A is orthogonally diagonalizable and invertible, then A^{-1} is also orthogonally diagonalizable

$$A = PDP^{-1} = PDP^T$$

$$A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1} D^{-1} P^{-1}$$

$$= P D^{-1} P^{-1}$$

also diagonal

e.g. $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}\right]$$

$$\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{matrix}}_{D^{-1}}$$

Spectral Decomposition

the set of eigenvalues of A is called the <u>spectrum</u> of A.

If A is orthogonally diagonalizable, then

$$A = PDP^T$$

$$= \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{u}_1^{\,T} \\ \vdots \\ \vec{u}_n^{\,T} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 \vec{u}_1 & \lambda_2 \vec{u}_2 & \cdots & \lambda_n \vec{u}_n \end{bmatrix} \begin{bmatrix} \vec{u}_1^{\,T} \\ \vdots \\ \vec{u}_n^{\,T} \end{bmatrix}$$

$$= \lambda_1 \vec{u}_1 \vec{u}_1^{\,T} + \lambda_2 \vec{u}_2 \vec{u}_2^{\,T} + \cdots + \lambda_n \vec{u}_n \vec{u}_n^{\,T}$$

if A is $n \times n$
then $\vec{u}_i$ is $n \times 1$
and $\vec{u}_i \vec{u}_i^{\,T}$ is $n \times n$

$$A\vec{x} = \underbrace{\lambda_1 \vec{u}_1 \vec{u}_1^T \vec{x}}_{\substack{\text{projection} \\ \text{of } \vec{x} \text{ onto} \\ \text{subspace} \\ \text{spanned by } \vec{u}_1}} + \lambda_2 \vec{u}_2 \vec{u}_2^T \vec{x} + \cdots + \lambda_n \vec{u}_n \vec{u}_n^T \vec{x}$$

__example__

$$A = \begin{bmatrix} 7 & 4 & -4 \\ 4 & 5 & 0 \\ -4 & 0 & 9 \end{bmatrix}$$

$$\lambda_1 = 13, \quad \vec{u}_1 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

$$\lambda_2 = 7, \quad \vec{u}_2 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$\lambda_3 = 1, \quad \vec{u}_3 = \begin{bmatrix} -2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$$

$$P = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix}$$

$$D = \begin{bmatrix} 13 & & 0 \\ & 7 & \\ 0 & & 1 \end{bmatrix}$$

$$A = PDP^T = \lambda_1 \vec{u}_1 \vec{u}_1^T + \lambda_2 \vec{u}_2 \vec{u}_2^T + \lambda_3 \vec{u}_3 \vec{u}_3^T$$

$$= 13 \begin{bmatrix} 4/9 & 2/9 & -4/9 \\ 2/9 & 1/9 & -2/9 \\ -4/9 & -2/9 & 4/9 \end{bmatrix} + 7 \begin{bmatrix} 1/9 & 2/9 & 2/9 \\ 2/9 & 4/9 & 4/9 \\ 2/9 & 4/9 & 4/9 \end{bmatrix} + 1 \begin{bmatrix} 4/9 & -4/9 & 2/9 \\ -4/9 & 4/9 & -2/9 \\ 2/9 & -2/9 & 1/9 \end{bmatrix}$$