Exam 2

- Friday, 7/13, 8:40-9:40 AM in EE 170
- Exam 2 will cover the following lessons:
  - 2.9, 3.1, 3.2, 3.3, 4.1, 4.2, 4.3, 4.5, 4.6, 5.1, 5.2, 5.3(1)
  - (HW 12-23)
- 8 multiple-choice, 4 worked-out, 10 true-or-false

Review

Basis : linearly ~~independingz~~ independent spanning set

can be used as coordinates in a subspace

for example, $\vec{b_1} = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$ $\vec{b_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are basis of subspace B of $\mathbb{R}^3$

$\vec{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$. in the subspace.

write the B-coordinates for $\vec{x}$

$\vec{x} = c_1 \vec{b_1} + c_2 \vec{b_2} \Rightarrow c_1 = ?,\ c_2 = ?$

$$\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad \left. \begin{matrix} c_1 = 2 \\ c_2 = 3 \end{matrix} \right\} \text{B-coords}$$

The # of basis vectors in a subspace is called the <u>dimension</u>. The <u>Rank</u> of a matrix is the dimension of the column space.

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}$$

$$\sim \cdots \sim \begin{bmatrix} \boxed{2} & 5 & -3 & -4 & 8 \\ 0 & \boxed{-3} & 2 & 5 & -7 \\ 0 & 0 & 0 & \boxed{4} & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rank $A = 3$

dim Col $A = 3$

dim Nul $A = 2$

$$\boxed{\text{dim Col } A + \text{dim Nul } A = \text{\# of columns}}$$

dim Row A = dim Col $A = 3$

determinant: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

3×3: two methods – cofactor expansion, Sarrus' rule

$$A = \begin{bmatrix} 1^{+} & 5^{-} & 0^{+} \\ 2^{-} & 4^{+} & -1^{-} \\ 0^{+} & -2^{-} & 0^{+} \end{bmatrix}$$

expand along any row or column, but best w/ row/column w/ most zeros

expand along row 3

$$\det(A) = 0 \begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + 0 \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}$$

$$= (2)(-1) = -2$$

$\det(A) \neq 0 \rightarrow A^{-1}$ exists

$$\left[\begin{matrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{matrix}\right] \begin{matrix} 1 & 5 \\ 2 & 4 \\ 0 & -2 \end{matrix}$$

$$\begin{matrix} - & - & - & + & + & + \end{matrix}$$

$$= 0 + 0 + 0 - 0 - (-2)(-1) - 0$$

$$= -2$$

4×4: cofactor expansion

Row replacements do not affect determinants

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$\begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix} = -1-(-2) = 1$$

each row swap changes sign

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$\det\left(\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}\right) = 1$$

$$\det\left(\begin{bmatrix} -3 & -3 \\ 2 & 1 \end{bmatrix}\right) = 3 = 3\det\left(\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}\right)$$

$$\det(AB) = \det(A)\det(B)$$

$$\det(A+B) \neq \det(A) + \det(B)$$

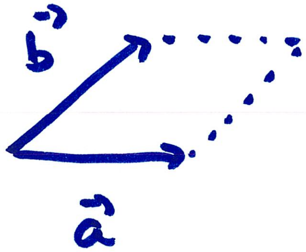
Cramer's Rule

$$3x_1 - 2x_2 = 6$$
$$-5x_1 + 4x_2 = 8$$

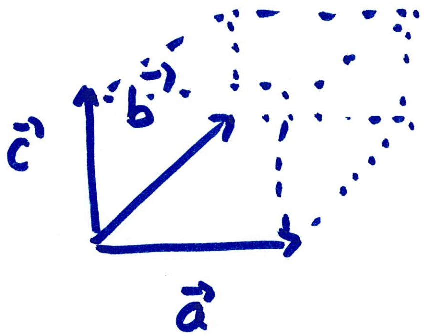
$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$x_1 = \frac{\det(A_1(\vec{b}))}{\det(A)} = \frac{\begin{vmatrix} 6 & -2 \\ 8 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -5 & 4 \end{vmatrix}} = \frac{40}{2} = 20$$

$$x_2 = \frac{\det(A_2(\vec{b}))}{\det(A)} = \frac{\begin{vmatrix} 3 & 6 \\ -5 & 8 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -5 & 4 \end{vmatrix}} = \frac{54}{2} = 27$$



$$\left|\det\left(\left[\vec{a}\ \ \vec{b}\right]\right)\right| = \text{area of parallelogram}$$



$$\left|\det\left(\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]\right)\right| = \text{volume of parallelopiped.}$$

No need to memorize the 10 axioms of a vector space

But you do need to know what a subspace is.

a) existence of $\vec{0}$

b) closed under addition

c) closed under scalar multiplication

$a\vec{v} + b\vec{u}$ is in subspace

linear combos of vectors in subspace remain in the subspace

A set of n linearly independent vectors in $\mathbb{R}^n$ is automatically a basis for $\mathbb{R}^n$

→ if columns of a matrix span $\mathbb{R}^n$, then $A\vec{x} = \vec{b}$ has a solution for any $\vec{b}$ in $\mathbb{R}^n$

Basis for Col A are pivot columns in the original matrix A.

Basis for Nul A are from solution of $A\vec{x} = \vec{0}$

Basis for Row A are from the nonzero rows of echelon form of A.

because row operations change linear dependence among the rows but NOT the columns.

If $A\vec{x} = \lambda \vec{x}$ ⇒ $\vec{x}$ is not rotated by A, only lengthened or shortened

then $\vec{x}$ is an eigenvector corresponding to the eigenvalue λ

eigenspace : subspace spanned by eigenvector(s)

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$A\vec{x} = \lambda\vec{x}$

$(A - \lambda I)\vec{x} = \vec{0}$

$\boxed{\text{eigenvector} \neq \vec{0}}$

nontrivial $\vec{x}$ if $\det(A - \lambda I) = 0$

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 4-\lambda & 1 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(4-\lambda)(1-\lambda) + 2\right] = 0$$

$$(1-\lambda)(\lambda^2 - 5\lambda + 6) = 0$$

$$(1-\lambda)(\lambda - 2)(\lambda - 3) = 0$$

$$\lambda = 1, 2, 3$$

eigenvector for $\lambda = 1$

$$(A - \lambda I)\vec{x} = \vec{0} \quad \begin{bmatrix} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$, x_2 free

$x_3 = 0$

$$\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

eigenvector for $\lambda = 2$

$$(A - \lambda I)\vec{x} = \vec{0} \quad \begin{bmatrix} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 free

$x_2 = x_3$ $\quad \vec{v} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

$x_1 = -\frac{1}{2} x_3$

choose $x_3 = 2$

eigenvector for $\lambda = 3$ is $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

this means A is diagonalizable : $A = PDP^{-1}$

$$P = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

If one λ is zero, then A^{-1} does not exist.