5.7 Applications to Differential Equations

HW 28+29 due together

basic differential eq: $x'(t) = a\,x(t)$ x: scalar function of t

solution: $x(t)$ that satisfies the D.E.

$x'(t) = a\,x(t)$

has solution $x(t) = Ce^{at}$ C: constant, a: const.

check: $x'(t) = C\cdot ae^{at} = a\cdot \underbrace{Ce^{at}}_{x(t)}$

now consider a system of first-order linear D.E's.

$$
\begin{aligned}
x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\vdots \\
x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n
\end{aligned}
$$

simple example:

$$
\begin{aligned}
x_1' &= x_1 + 2x_2 \\
x_2' &= 3x_1 + 4x_2
\end{aligned}
$$

$$\begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{n2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\vec{x}' = A\vec{x}$$

e.g. $x_1' = x_1 + 2x_2$
$x_2' = 3x_1 + 4x_2$ $\Rightarrow$ $\vec{x}' = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \vec{x}$

$\rightarrow$ how to solve this?

$$\vec{x} = ?$$

Simple case: $x_1' = x_1$

$x_2' = 2x_2$ $\Rightarrow$ $\vec{x}\,' = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \vec{x}$

Solve by using calculus: $x_1 = c_1 e^t$

$x_2 = c_2 \cdot e^{2t}$

as vector equation:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \underbrace{c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

fundamental solutions

(or eigenfunctions)

each is a solution of $\vec{x}\,' = A\vec{x}$

and a linear combination of them

is also a solution.

each solution is of ~~them~~ the form $\vec{x} = e^{\lambda t} \vec{v}$

what are λ and $\vec{v}$?

$\vec{x}' = A\vec{x}$ solution: $\vec{x} = e^{\lambda t}\vec{v}$

then $\vec{x}' = \lambda e^{\lambda t}\vec{v}$

$\lambda \cancel{e^{\lambda t}} \vec{v} = A \cancel{e^{\lambda t}} \vec{v}$ $e^{\lambda t} \neq 0$

$\boxed{\lambda \vec{v} = A\vec{v}}$ so $\lambda, \vec{v}$ are the eigenvalue/eigenvector pair of A.

If A is 2×2, there are 2 fundamental solutions

" " $n \times n$ " " n " "

each is $e^{\lambda t}\vec{v}$

cases: λ's are distinct

λ's are complex

λ's repeated (we won't look at this in 5.7)

~~ex~~ <u>example</u> $\vec{x}' = \underbrace{\begin{bmatrix} 9 & -2 \\ 6 & 1 \end{bmatrix}}_{A} \vec{x}$ $\quad$ $x_1' = 9x_1 - 2x_2$

$x_2' = 6x_1 + x_2$

$\lambda = 3, \quad 7$

$\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2 fundamental solutions: $\vec{x}_1 = e^{3t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\vec{x}_2 = e^{7t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

general solution:

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2 = c_1 e^{3t}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{7t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

c_1, c_2 come from <u>initial conditions</u> e.g. $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\ldots \quad c_1 = -\frac{1}{2} \quad c_2 = \frac{3}{2}$

note $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is also a solution

the origin $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is known as an equilibrium solution

solutions will move away from the origin if both λ's are positive, the origin is a <u>source</u> or a <u>repeller</u>

solutions will move toward origin if both λ's are negative $\rightarrow$ <u>sink</u> or <u>attractor</u>

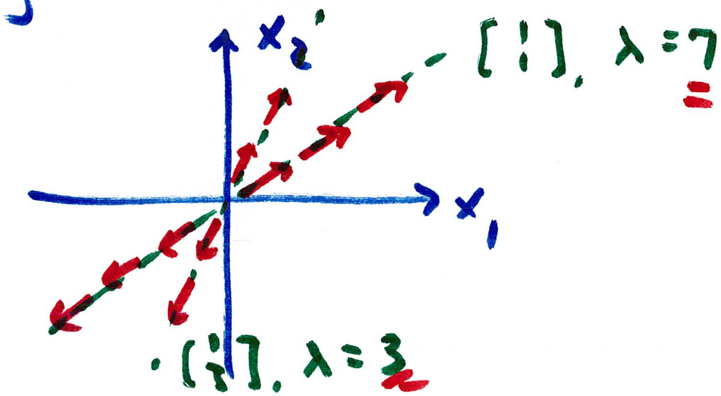
if λ's are of mixed signs, the origin is a <u>saddle point</u> toward $\vec{0}$ in some directions, away in others

in this example, $\lambda = 3$, $\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\lambda = 7$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

arrow away if $\lambda > 0$

" toward $\lambda < 0$



complex λ's : solutions are spirals

<u>example</u>

$$\vec{x}' = \begin{bmatrix} -8 & 10 \\ -1 & -2 \end{bmatrix} \vec{x}$$

$$\lambda = -5+i \quad , \quad -5-i$$

$$\vec{v} = \begin{bmatrix} 3-i \\ 1 \end{bmatrix} \quad , \quad \begin{bmatrix} 3+i \\ 1 \end{bmatrix}$$

fundamental solutions:

$$\left.\begin{aligned} \vec{x}_1 &= e^{(-5+i)t} \begin{bmatrix} 3-i \\ 1 \end{bmatrix} \\ \vec{x}_2 &= e^{(-5-i)t} \begin{bmatrix} 3+i \\ 1 \end{bmatrix} \end{aligned}\right\} \text{complex-valued}$$

general solution

real-valued

$$\vec{x} = \underbrace{c_1}_{\text{complex-valued}} e^{(-5+i)t} \begin{bmatrix} 3-i \\ 1 \end{bmatrix} + \underbrace{c_2}_{\text{complex-valued}} e^{(-5-i)t} \begin{bmatrix} 3+i \\ 1 \end{bmatrix}$$

this solution is inconvenient in many applications

need real-valued equivalent

$$\vec{x}_1 = e^{(-5+i)t}\begin{bmatrix} 3-i \\ 1 \end{bmatrix}$$

$$= e^{-5t}e^{it}\begin{bmatrix} 3-i \\ 1 \end{bmatrix} \qquad e^{it} = \cos t + i\sin t$$

$$= e^{-5t}(\cos t + i\sin t)\begin{bmatrix} 3-i \\ 1 \end{bmatrix}$$

$$= e^{-5t}\begin{bmatrix} 3\cos t + \sin t - i\cos t + 3i\sin t \\ \cos t + i\sin t \end{bmatrix}$$

$$\vec{x}_1 = e^{-5t}\left(\begin{bmatrix} 3\cos t + \sin t \\ \cos t \end{bmatrix} + i\begin{bmatrix} 3\sin t - \cos t \\ \sin t \end{bmatrix}\right)$$

repeat w/ $\vec{x}_2$

$$\vec{x}_2 = e^{-5t}\left(\begin{bmatrix} 3\cos t + \sin t \\ \cos t \end{bmatrix} - i\begin{bmatrix} 3\sin t - \cos t \\ \sin t \end{bmatrix}\right)$$

$$\left.\begin{aligned}
\text{let } \vec{u} &= Re(\vec{x}_1) = e^{-5t}\begin{bmatrix} 3\cos t + \sin t \\ \cos t \end{bmatrix} \\
\vec{v} &= \cancel{Re}\ Im(\vec{x}_1) = e^{-5t}\begin{bmatrix} 3\sin t - \cos t \\ \sin t \end{bmatrix}
\end{aligned}\right\} \text{real-valued}$$

~~so the gener~~

each is now a real-valued fundamental solution

general solution:

$$\vec{x} = c_1 e^{-5t}\begin{bmatrix} 3\cos t + \sin t \\ \cos t \end{bmatrix} + c_2 e^{-5t}\begin{bmatrix} 3\sin t - \cos t \\ \sin t \end{bmatrix}$$

everything is real

if $\lambda = a \pm ib$ if a $(Re(\lambda))$ is positive,

solutions spiral <u>away</u> from $\vec{0}$

if a $(Re(\lambda))$ is negative

solutions spiral <u>into</u> $\vec{0}$

3x3

$$A = \begin{bmatrix} -3 & -10 & 0 \\ 6 & 5 & 6 \\ -1 & 7 & -4 \end{bmatrix}$$

$\lambda = -3, -1, 2$

$\vec{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix}$

$$\vec{x} = c_1 e^{-3t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -5 \\ 1 \\ 4 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix}$$

$\rightarrow$ mixed signs, so $\vec{0}$ is a saddle point

if $c_3 = 0$, solutions go to $\vec{0}$ as $t \to \infty$

if $c_1 = c_2 = 0$, solutions go away from $\vec{0}$ as $t \to \infty$

decoupling differential eqs.

$$\vec{x}\,' = \underbrace{\begin{bmatrix} 9 & -2 \\ 6 & 1 \end{bmatrix}}_{A} \vec{x}$$

$x_1' = 9x_1 - 2x_2$

$x_2' = 6x_1 + x_2$

"coupled" because x_i' depends on other x's

these can be decoupled by diagonalizing A

$$\vec{x}\,' = A\vec{x}$$

$$= PDP^{-1}\vec{x}$$

let $\vec{y} = P^{-1}\vec{x}$

so $P\vec{y} = \vec{x}$

$P\vec{y}\,' = \vec{x}\,'$, $\vec{y}\,' = P^{-1}\vec{x}\,'$

$$\vec{x}\,' = PD\vec{y}$$

$$P^{-1}\vec{x}\,' = P^{-1}PD\vec{y}$$

$$\vec{y}\,' = D\vec{y}$$

D diagonal

$\vec{y}$ is a decoupled sys.

solve for $\vec{y}$, then $\vec{x} = P\vec{y}$