Math 265 Summer 2018 - Quiz 09

7/20/2018 - 10 Minutes **Name:**

Remember that your work is graded on the quality of your writing and explanation as well as the validity of the mathematics.

(1) (10 Points) For the following matrix

$$A = \begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix}$$

1) find an eigenvalue λ and a corresponding eigenvector $\mathbf{v}$.

2) find an invertible matrix P and a matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that given matrix has the form $A = PCP^{-1}$.

Answer:

1) The characteristic polynomial of A is $\det(A - \lambda I) = \lambda^2 - 4\lambda + 8 = 0$, so we get the eigenvalues $\lambda_1 = 2 + 2i, \lambda_2 = 2 - 2i$.

Solve the equations

$$\begin{cases} x_2 = \lambda_i x_1 \\ -8x_1 + 4x_2 = \lambda_i x_2 \end{cases} , \quad \text{for } i = 1, 2$$

We get the corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2+2i \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2-2i \end{bmatrix}$.

2) We can use λ_1 and $\mathbf{v_1}$ to constructe P and C:

$$P = [\mathrm{Re}\mathbf{v_1} \ \mathrm{Im}\mathbf{v_1}] = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$$

and

$$\begin{cases} a = \mathrm{Re}\lambda_1 = 2 \\ b = -\mathrm{Im}\lambda_1 = -2 \end{cases} \Rightarrow C = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$$

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(2) (10 Points) Let A be a 2×2 matrix with eigenvalues -3 and 1 and corresponding eigenvectors $\mathbf{v_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let $\mathbf{x}(t)$ be the position of a particle at time t, solve the initial value problem $\mathbf{x}' = A\mathbf{x}, \mathbf{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Answer: The solution of given problem should be in the following form

$$\mathbf{x}(t) = c_1\mathbf{v_1}e^{\lambda_1 t} + c_2\mathbf{v_2}e^{\lambda_2 t} = c_1 \begin{bmatrix} -e^{-3t} \\ e^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} e^t \\ e^t \end{bmatrix} = \begin{bmatrix} -c_1 e^{-3t} + c_2 e^t \\ c_1 e^{-3t} + c_2 e^t \end{bmatrix}$$

notice that the initial condition give us:

$$\begin{cases} -c_1 + c_2 = 2 \\ c_1 + c_2 = 3 \end{cases} \Rightarrow \begin{cases} c_1 = 1/2 \\ c_2 = 5/2 \end{cases}$$

Therefore the solution is:

$$\mathbf{x}(t) = \frac{1}{2}\begin{bmatrix} -e^{-3t} \\ e^{-3t} \end{bmatrix} + \frac{5}{2}\begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix}$$

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