2.8 Subspaces of $\mathbb{R}^n$

Subspace : collection of vectors in some subset of $\mathbb{R}^n$

for example, vectors of the form $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ are in a

subspace of $\mathbb{R}^3$

here, the space is $\mathbb{R}^2$

A subspace must have these properties:

a) It must contain the zero vector ($\vec{0}$)

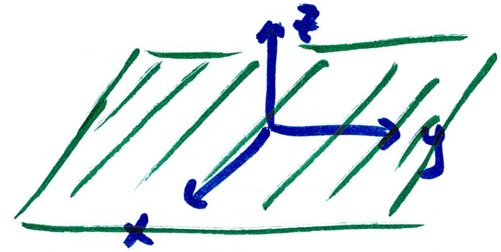
b) It must be <u>closed under addition</u>

for each $\vec{u}$ and $\vec{v}$ in the set, $\vec{u}+\vec{v}$ is also in the set.

c) It must be <u>closed under multiplication</u>

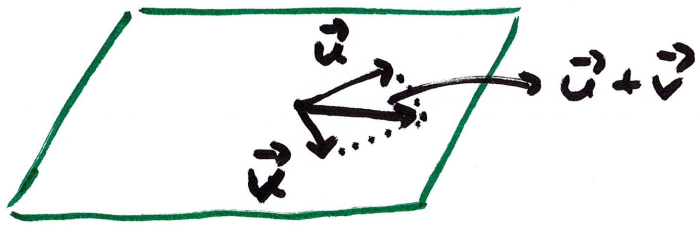
for each $\vec{u}$ in the set, $c\vec{u}$ must also be in the set.

all vectors of the form $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$. Are the requirements met?



a) is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ in the set? Yes

b). closed under addition?



Yes, ANY vectors $\begin{bmatrix} x_1 \\ y_1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \\ 0 \end{bmatrix}$ is always have a sum $\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 0 \end{bmatrix}$ that's on xy plane.

c). closed under multiplication?

$$c \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} cx \\ cy \\ 0 \end{bmatrix}$$ still on xy plane

so, yes.

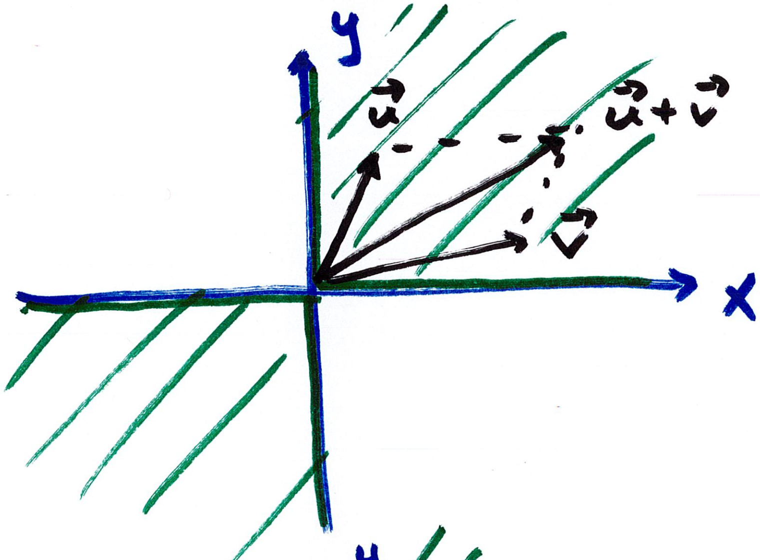
therefore, $\mathbb{R}^2$ (xy plane) $\underline{\underline{\text{is}}}$ a subspace.

example Is this a subspace?

All vectors (x, y) $\left(\text{or } \begin{bmatrix} x \\ y \end{bmatrix}\right)$ such that

$x \geq 0, y \geq 0$ and $x \leq 0, y \leq 0$

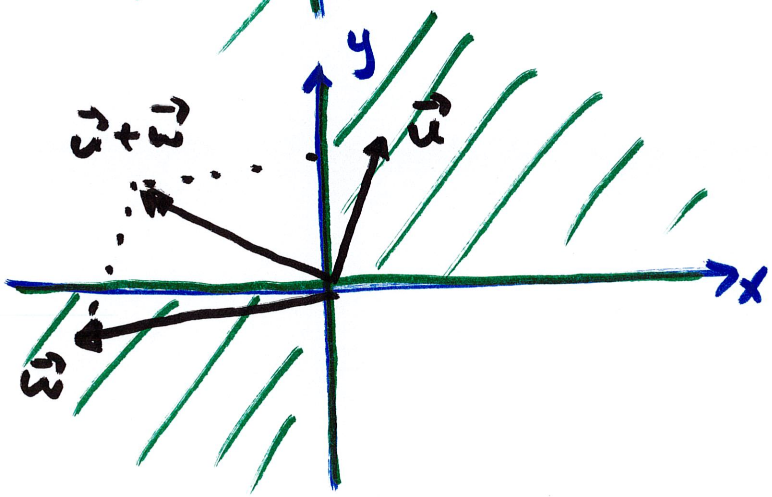
All (x, y) such that x and y have same sign or ~~both zero~~ one or both zero.



a) is $\vec{0}$ in this set? Yes

b) closed under addition?
No, because $\vec{u}+\vec{w}$ is in 2nd quadrant.

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$



c) closed under multiplication?
~~N~~ Yes

There, this set is NOT a subspace.

Some notable subspaces connected to a matrix A.

column space - all linear combos of columns of A

null space - all solutions of $A\vec{x} = \vec{0}$

example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & -1 & -2 \end{bmatrix}$

column space of A : $\text{Col } A = \text{span}\left\{ \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ -2 \end{bmatrix} \right\}$

Is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ in Col A? $\Rightarrow$ Solve $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

no solution, so no x_1, x_2, x_3 exist such that

$$x_1 \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

therefore, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is <u>NOT</u> in Col A.

null space of A : Nul A

solutions of $A\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 0 & -1 & -2 & 0 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 is free $\quad x_2 = -2x_3 \quad x_1 = x_3$

so $\vec{x} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$\text{Nul } A = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$

is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ in Nul A ? No

equivalent question: $\cancel{Ax}$ is $A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \vec{0}$

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\text{Col } A = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

the last one, being a linear combo of first 3, is not needed to describe Col A.

the <u>minimum</u> set of vectors needed to describe a subspace is called a <u>basis</u>

the vectors in a basis (called bases) are <u>linearly independent</u>.

example

$$A = \begin{bmatrix} 3 & 6 & 21 & -24 \\ 5 & 9 & 29 & -34 \\ 2 & 3 & 8 & -10 \end{bmatrix}$$

find a basis for Nul A

so reduce $\begin{bmatrix} A & \vec{0} \end{bmatrix} = \begin{bmatrix} 3 & 6 & 21 & -24 & 0 \\ 5 & 9 & 29 & -34 & 0 \\ 2 & 3 & 8 & -10 & 0 \end{bmatrix}$

$$\sim \cdots \sim \begin{bmatrix} 1 & 3 & 13 & -14 & 0 \\ 0 & 1 & 6 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3, x_4 are free

$$x_2 = \underbrace{-6x_3 + 6x_4}$$

$$x_1 = -3x_2 - 13x_3 + 14x_4$$

$$= 5x_3 - 4x_4$$

$$\vec{x} = \begin{bmatrix} 5x_3 - 4x_4 \\ -6x_3 + 6x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -6 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

basis of Nul A is $\left\{ \begin{bmatrix} 5 \\ -6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \\ 0 \\ 1 \end{bmatrix} \right\}$

basis of Col A?

$$A = \begin{bmatrix} 3 & 6 & 21 & -24 \\ 5 & 9 & 29 & -34 \\ 2 & 3 & 8 & -10 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \boxed{1} & 3 & 13 & -14 \\ 0 & \boxed{1} & 6 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

the columns in A that correspond to the pivot columns are basis vectors of Col A

Col A has basis $\left\{ \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 9 \\ 3 \end{bmatrix} \right\}$

why? ~~b~~ basis vectors are linearly independent

→ solving $[A \ \ \vec{0}]$ this identifies the dependence relationship (and thus which are linearly indp)

this method works because ERO's do not change dependency of columns.

basis of Col A is related to pivots.

If an $n \times n$ matrix has n pivots then the columns of A are basis vectors of Col A

→ columns span $\mathbb{R}^n$, $\text{Col } A = \mathbb{R}^n$

→ A is invertible

→ $A\vec{x} = \vec{0}$ only has trivial solution