Exam 1

- Location: EE170
- Time: 8:40am-9:40am
  - Please arrive 5-10 minutes early
- 7 multiple-choice problems
- 4 worked-out problems
- 9 true-or-false problems

Review

$$x_1 - 8x_3 = -3$$

$$x_2 - x_3 = -1$$

augmented matrix
$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\left[\begin{array}{cccc} 1 & 0 & -8 & -3 \\ 0 & 1 & -1 & -1 \end{array}\right]$$

already in reduced row echelon form (pivot in each row)

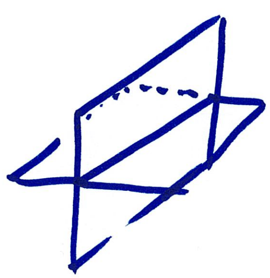
x_1, x_2 are <u>basic ~~fe~~ variables</u>

x_3 is <u>free</u>

$$x_2 - x_3 = -1 \rightarrow x_2 = x_3 - 1$$

$$x_1 - 8x_3 = -3 \rightarrow x_1 = 8x_3 - 3$$

$$\vec{x} = \begin{bmatrix} 8x_3 - 3 \\ x_3 - 1 \\ x_3 \end{bmatrix} = \underbrace{x_3 \begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \\ 0 \end{bmatrix}}_{\text{line}}$$



no solution if inconsistent
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$
(augmented matrix)

system has no solution

infinitely-many if zero row
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

unique solution if every variable gets a pivot
$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$x_1 - 8x_3 = -3$$
$$x_2 - x_3 = -1$$

is equivalent to
$$\begin{bmatrix} 1 & 0 & -8 & -3 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

and
$$\begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix} \rightarrow A\vec{x} = \vec{b}$$

and $\underbrace{x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -8 \\ -1 \end{bmatrix}}_{\text{linear combination}} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$

finding solution is the same as asking if $\begin{bmatrix} -3 \\ -1 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -8 \\ -1 \end{bmatrix}$

Is $\vec{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ a linear combo of

$\vec{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{bmatrix}$$

two identical rows means one is zero row $\Rightarrow$ at least one solution (x_1, x_2, x_3)

Such that $x_1\begin{bmatrix}1\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\\2\end{bmatrix}+x_3\begin{bmatrix}5\\4\\8\end{bmatrix}=\begin{bmatrix}2\\3\\6\end{bmatrix}$

this also means $\vec{b}$ is in span$\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$

If <u>any</u> vector in $\mathbb{R}^n$ can be written as a linear combo of $\vec{a}_1, \vec{a}_2, \ldots,$ then $\vec{a}_1, \vec{a}_2, \ldots$ span $\mathbb{R}^n$

Do $\begin{bmatrix}0\\0\\-2\end{bmatrix}, \begin{bmatrix}0\\-3\\8\end{bmatrix}, \begin{bmatrix}4\\-1\\-5\end{bmatrix}$ Span $\mathbb{R}^3$

$\Rightarrow$ does $x_1\begin{bmatrix}0\\0\\-2\end{bmatrix}+x_2\begin{bmatrix}0\\-3\\8\end{bmatrix}+x_3\begin{bmatrix}4\\-1\\-5\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$

have a solution for <u>any</u> b_1, b_2, b_3?

$\Rightarrow$ ~~So does~~ is $\begin{bmatrix}0 & 0 & 4 & b_1\\0 & -3 & -1 & b_2\\-2 & 8 & -5 & b_3\end{bmatrix}$ consistent for any b_1, b_2, b_3?

if one pivot in every column of the coefficient matrix, then there is a solution for any b_1, b_2, b_3

$$\begin{bmatrix} \boxed{-2} & 8 & -5 & b_3 \\ 0 & \boxed{-3} & -1 & b_2 \\ 0 & 0 & \boxed{4} & b_1 \end{bmatrix}$$

3 pivots not in last col.
3 variables
a unique solution exists for any b_1, b_2, b_3

yes, the vectors do span $\mathbb{R}^3$

$A\vec{x} = \vec{0}$ (homogeneous system) ALWAYS has a solution (the trivial solution $\vec{0}$)

$A\vec{x} = \vec{0}$ has nontrivial solution if there is a free variable

$A\vec{x} = \vec{b}$, if <u>consistent</u>, has a solution that is a shifted version of that of $A\vec{x} = \vec{0}$.

linear independence

$\vec{a}_1, \vec{a}_2, \vec{a}_3, \dots$ are linearly independent if

$x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 + \cdots = \vec{0}$ means $x_1 = x_2 = x_3 = \cdots = 0$
and is the <u>only</u> solution
to $[\vec{a}_1\ \vec{a}_2\ \cdots]\vec{x} = \vec{0}$

this means $\left[\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_n \,\middle|\, \vec{0}\right]$ has pivot in every column of coefficient matrix.

An $n \times m$ matrix defines a linear transformation

$T(\vec{x}) = A\vec{x}$, $T: \mathbb{R}^m \to \mathbb{R}^n$

e.g. A is 3×5
$T: \mathbb{R}^5 \to \mathbb{R}^3$

$$\underset{3\times 5}{\begin{bmatrix}\vdots&\vdots&\vdots&\vdots&\vdots\end{bmatrix}}\underset{5\times 1}{\begin{bmatrix}\vdots\end{bmatrix}} = \underset{3\times 1}{\begin{bmatrix}\vdots\end{bmatrix}}$$

Transformation is <u>onto</u> $\mathbb{R}^n$ if every vector in $\mathbb{R}^n$ is an image of at <u>least</u> one vector in the domain of T.

(the solution to $A\vec{x} = \vec{b}$ exists for any $\vec{b}$)

Transformation is <u>one-to-one</u> if ~~any~~ a vector in $\mathbb{R}^n$ is an image of at <u>most</u> one vector in the domain of T

(uniqueness of solution)

$$A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix} \quad C = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$$

$$A + 2B = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 7 \\ 5 & 13 & 16 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 4 & -1 \\ 0 & 3 \\ 5 & 2 \end{bmatrix}$$

AC does not ~~to~~ exist because

$$\begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$$

$$2 \times \boxed{3} \qquad \boxed{2} \times 2$$

must match

CA is defined: $\begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 11 & -9 & 4 \\ -1 & 3 & 2 \end{bmatrix}$$

$$C^2 = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -9 \\ 0 & 1 \end{bmatrix}$$

$$C^{-1} = \frac{1}{(2)(1)-(0)(-3)} \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\underbrace{ad-bc}_{\text{determinant}}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$ inverse ?

$$\left[\begin{array}{ccc:ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right]$$

then do ERO's until left half is I

Subspaces properties

1) contains $\vec{0}$

2) closed under addition → $\vec{u}, \vec{v}$ is ~~op~~ subspace, then $\vec{u}+\vec{v}$ is also in subspace

3) closed under scalar multiplication → $c\vec{u}$ is in subspace

Column space of A : linear combos of A

Nullspace of A : solutions of $A\vec{x} = \vec{0}$

basis: minimum ~~# of~~ set of vectors to span subspace
(linearly independent → count pivots)