1.9 The Matrix of Linear Transformation

$T: \mathbb{R}^m \to \mathbb{R}^n \qquad T(\vec{x}) = A\vec{x}$

What is the minimum we need to know about T to find A? If $\cancel{T(\vec{x})} = T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 13 \\ 13 \end{bmatrix}$, $A = ?$

It turns out we just need to know the transformation of the standard unit vectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_m$

$T(\vec{e}_1), T(\vec{e}_2), \dots, T(\vec{e}_m) \Rightarrow$ then we know A

$$\mathbb{R}^3: \quad \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(sometimes written $\vec{e}_1 = (1, 0, 0)$, $\vec{e}_2 = (0, 1, 0)$, etc)

$\vec{e}_i$'s are columns of identity matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Suppose we know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}3\\2\end{bmatrix}$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}-1\\-2\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}4\\5\end{bmatrix}$$

what is A such that $T(\vec{x})=A\vec{x}$?

since $\vec{e_1}, \vec{e_2}, \vec{e_3}$ span $\mathbb{R}^3$, ANY vector in $\mathbb{R}^3$ is a linear combo of $\vec{e_1}, \vec{e_2}, \vec{e_3}$

$\vec{b} = x_1\vec{e_1} + x_2\vec{e_2} + x_3\vec{e_3}$ for ANY $\vec{b}$ in $\mathbb{R}^3$

because T is a <u>linear</u> transformation, $T(\vec{u}+\vec{v})=T(\vec{u})+T(\vec{v})$

$T(c\vec{u})=cT(\vec{u})$

$$\begin{aligned}T(\vec{b}) &= T(x_1\vec{e_1}+x_2\vec{e_2}+x_3\vec{e_3})\\ &= T(x_1\vec{e_1})+T(x_2\vec{e_2})+T(x_3\vec{e_3})\\ &= x_1T(\vec{e_1})+x_2T(\vec{e_2})+x_3T(\vec{e_3})\end{aligned}$$

$$= \underbrace{\begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}}$$

$$= \underbrace{\begin{bmatrix} 3 & -1 & 4 \\ 2 & -2 & 5 \end{bmatrix}}_{A} \vec{x}$$

$A \rightarrow$ "standard matrix of the linear transformation T"

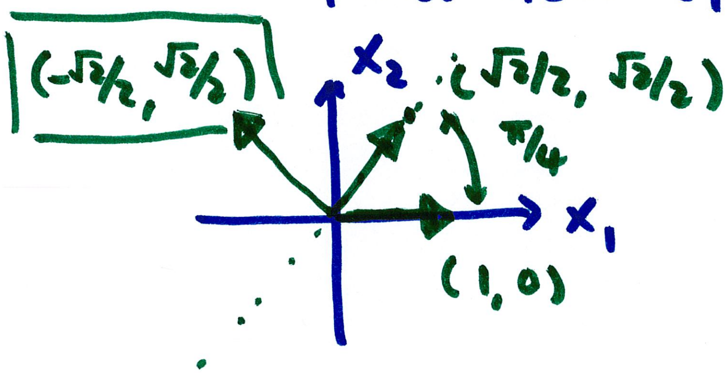
$$\boxed{\begin{array}{c} A \text{ is the matrix whose columns are} \\ T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_m) \end{array}}$$

Some example transformations

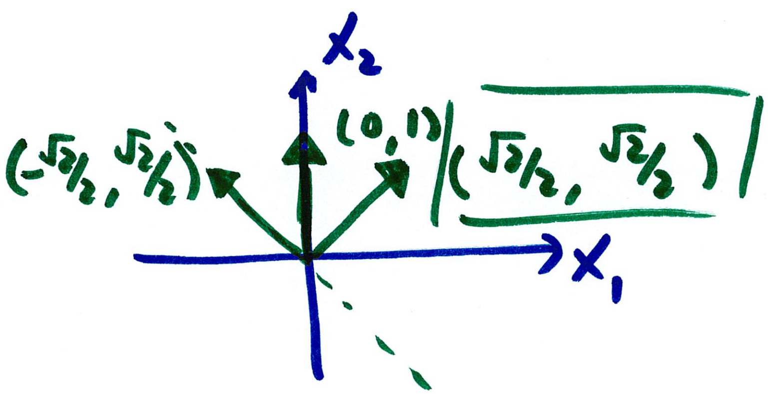
<u>example</u> $T: \mathbb{R}^2 \to \mathbb{R}^2$ T rotates a vector about the origin by $\frac{\pi}{4}$ radians, then reflects about vertical (x_2) axis

Find A.

find how $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are transformed.



$$T(\vec{e_1}) = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$



$$T(\vec{e_2}) = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}}$$

see p. 74-76 for some standard transformation matrices
reflection, contraction/expansion, shear, projection

A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be <u>onto</u> $\mathbb{R}^m$ if each $\vec{b}$ in $\mathbb{R}^m$ is the image of <u>at least one</u> $\vec{x}$ in $\mathbb{R}^n$

another way to say this: for any $\vec{b}$ in $\mathbb{R}^m$ there is a solution to $A\vec{x} = \vec{b}$

<u>example</u>

$$A = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 2 & 4 & -1 \\ 3 & 5 & 2 \end{bmatrix}$$

$T: \mathbb{R}^3 \to \mathbb{R}^4$

Is T onto $\mathbb{R}^4$

equivalent question: is there at least one solution to $A\vec{x} = \vec{b}$ $\vec{b}$ is some vector in $\mathbb{R}^4$?

augmented matrix: $\begin{bmatrix} 1 & 0 & 5 & b_1 \\ 0 & 1 & 0 & b_2 \\ 2 & 4 & -1 & b_3 \\ 3 & 5 & 2 & b_4 \end{bmatrix}$

note this matrix can only have up to 3 pivot positions

so one of these rows will look like

$$\begin{bmatrix} 0 & 0 & 0 & \underbrace{\cdots\cdots}_{\text{some expression involving } b_1, b_2, b_3, b_4} \end{bmatrix}$$

if last element $\neq 0$ (which can happen for arbitrary b_1, b_2, b_3, b_4)

then there is no solution

so T is <u>NOT</u> <u>onto</u> $\mathbb{R}^4$

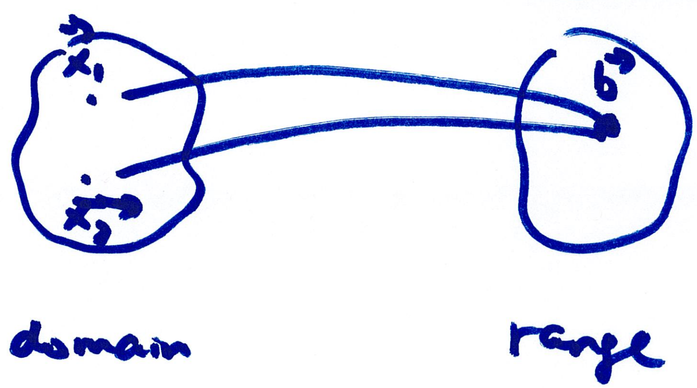
example $A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

$T: \mathbb{R}^4 \to \mathbb{R}^3$ onto $\mathbb{R}^3$

3 pivots, ~~can't have zero row~~, so there is <u>always</u> a solution to $A\vec{x} = \vec{b}$

so, yes, T is onto $\mathbb{R}^3$

If $T: \mathbb{R}^n \to \mathbb{R}^m$, T is <u>one-to-one</u> if each $\vec{b}$ in $\mathbb{R}^m$ is the image of <u>at most one</u> $\vec{x}$ in $\mathbb{R}^n$



this is NOT <u>one-to-one</u>

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$T: \mathbb{R}^4 \to \mathbb{R}^3$

is $T(\vec{x}) = A\vec{x}$ one-to-one?

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\left[\begin{array}{ccccc} 1 & -4 & 8 & 1 & b_1 \\ 0 & 2 & -1 & 3 & b_2 \\ 0 & 0 & 0 & 5 & b_3 \end{array}\right]$$

4 variables, 3 pivots, one free $\Rightarrow$ No unique solution

<u>not</u> <u>one-to-one</u>

$\boxed{\text{If } T: \mathbb{R}^n \to \mathbb{R}^m \text{ is one-to-one then } T(\vec{x}) = A\vec{x} = \vec{0} \text{ only has the trivial solution}}$

why? Recall the solution of $T(\vec{x}) = A\vec{x} = \vec{b}$ is a shifted version of $A\vec{x} = \vec{0}$, so if $A\vec{x} = \vec{0}$ has unique solution, the $A\vec{x} = \vec{b}$ can have <u>at most one solution</u>

Now we can summarize everything

$T: \mathbb{R}^n \to \mathbb{R}^m$ is <u>onto</u> $\mathbb{R}^m$ if columns of A span $\mathbb{R}^m$

$T: \mathbb{R}^n \to \mathbb{R}^m$ is <u>one-to-one</u> if columns of A are <u>linearly independent</u>