4.3 Linear Independence Sets and Bases

$\{\vec{v}_1, \dots, \vec{v}_p\}$ in V is linearly independent if

$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0}$ has only the trivial solution: $c_1 = c_2 = \cdots = c_p = 0$

very straight forward if vectors are in $\mathbb{R}^n$ ($A\vec{x} = \vec{0}$)

An alternate definition (for vectors typically not in $\mathbb{R}^n$)

→ An indexed set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ of two or more vectors with $\vec{v}_1 \neq \vec{0}$ is linearly <u>dependent</u> if and only if some $\vec{v}_j$ $(j > 1)$ is a linear combination of the preceding vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{j-1}$.

<u>example</u> $\{\underset{\vec{v}_1}{1}, \underset{\vec{v}_2}{t}, \underset{\vec{v}_3}{t^2}\}$ is linearly independent because $\vec{v}_j$ is never a linear combo of vectors preceding it.

what does it mean for a set to be a basis of some subspace H of a vector space V?

i) set is linearly independent $\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_p\}$

ii) the subspace spanned by the set must coincide with H

$$\rightarrow H = \text{span}\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_p\}$$

for example, $H = \mathbb{R}^3$

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$ is clearly linearly independent

but <u>not</u> a basis of $\mathbb{R}^3$ because ~~B is not~~ the vectors in B do not span $\mathbb{R}^3$

How about $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ -2 \\ -6 \end{bmatrix} \right\}$? Basis for $\mathbb{R}^3$?

independent? $\begin{bmatrix} 1 & 2 & -6 \\ 0 & 1 & -2 \\ 2 & 1 & -6 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \boxed{1} & 2 & -6 \\ 0 & \boxed{1} & -2 \\ 0 & -3 & 6 \end{bmatrix}$

2 pivots, no unique solution to $A\vec{x} = \vec{0}$

NOT linearly independent

so NOT basis for $\mathbb{R}^3$

$$B = \left\{ \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -4 \end{bmatrix} \right\}$$

indp? $\begin{bmatrix} 1 & 0 & 3 & 0 \\ -5 & 3 & -7 & 4 \\ 3 & -1 & 6 & -4 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 8 \end{bmatrix}$

4^{th} vector is a linear combo of the first 3

but B does span $\mathbb{R}^3$

can we make B a basis for $\mathbb{R}^3$?

yes, simply throw the 4^{th} (the one dependent on others) out.

$$C = \left\{ \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \\ 6 \end{bmatrix} \right\}$$ is basis for $\mathbb{R}^3$

The Spanning Set Theorem

$S = \{\vec{v}_1, \dots, \vec{v}_p\}$ in V, $H = \text{span}\{\vec{v}_1, \dots, \vec{v}_p\}$

a). If one of the vectors in S, for example, $\vec{v}_k$, is a linear combination of the remaining vectors in S, then the set formed by removing $\vec{v}_k$ from S still spans H.

b) If $H \neq \{\vec{0}\}$, some subset of S is a basis for H.

Finding basis for Nul A is easy.

- the standard way of solving $A\vec{x} = \vec{0}$ always gives us the basis for Nul A

<u>example</u> $A = \begin{bmatrix} -2 & 6 & -2 & -6 \\ 2 & -9 & -2 & 1 \\ -3 & 12 & 1 & -4 \end{bmatrix}$

$$[A \ \vec{0}] \sim \cdots \sim \begin{bmatrix} 1 & 0 & 5 & 8 & 0 \\ 0 & 3 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3, x_4 free

$x_1 = -5x_3 - 8x_4$

$x_2 = -4/3\, x_3 - 5/3\, x_4$

$$\text{Nul } A = x_3 \begin{bmatrix} -5 \\ -4/3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -8 \\ -5/3 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{basis for Nul } A = \left\{ \begin{bmatrix} -5 \\ -4/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -8 \\ -5/3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

basis for Col A are pivot columns in the <u>original</u> A (NOT the reduced form)

here, basis for Col A $= \left\{ \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 6 \\ -9 \\ 12 \end{bmatrix} \right\}$

There are two ways to look at a basis

1) ~~the~~ a minimum or the most efficient spanning set

2) ~~the~~ a largest possible linearly independent set for that subspace

<u>example</u> Find a basis for the set of vectors in $\mathbb{R}^2$ on the line $y = 5x$

rewrite this as a homogeneous "system" and find the nullspace.

$$-5x + y = 0$$

$$\underbrace{\begin{bmatrix} -5 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

Nul A ?

$$\begin{bmatrix} -5 & 1 & 0 \end{bmatrix}$$

pivot column free variable

y is free

$x = \frac{1}{5} y$

$$\begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} 1/5 \\ 1 \end{bmatrix}$$

so the basis is $\begin{bmatrix} 1/5 \\ 1 \end{bmatrix}$