4.2 Null space, Column space, and Linear Transformations

nullspace of a matrix A is set of all solutions to $A\vec{x} = \vec{0}$

$$\text{Nul } A = \{ \vec{x} : \vec{x} \text{ is in } \mathbb{R}^n \text{ and } A\vec{x} = \vec{0} \}$$

If A is $m \times n$ then Nul A is set of all vectors

that are from $\mathbb{R}^n$ to $\mathbb{R}^m$

$\vec{0}$ in

$A\vec{x} = \vec{0}$ defines null space implicitly. To ~~do~~ solve explicitly,

solve $A\vec{x} = \vec{0}$

Is null space of A a subspace?

a). $\vec{0}$ exists? Yes, because $A\vec{0} = \vec{0}$

b). Closed under addition? For $\vec{u}, \vec{v}$ such that $A\vec{u} = \vec{0}$, $A\vec{v} = \vec{0}$.

$A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \vec{0}$. Yes, closed under addition.

c). Closed under scalar multiplication? If $A\vec{u} = \vec{0}$, then

$A(c\vec{u}) = cA\vec{u} = \vec{0}$. Yes, closed.

An example of null space not related to matrix.

$\underbrace{y'' + 4y = 0}$ linear 2nd-order homogeneous differential eq.

solution : $y(x) = ?$

can be viewed as a transformation of a "vector" $y(x)$ such that it gets mapped to the zero "vector"

domain : all twice-differentiable functions

range : " " " " such that

$$y'' + 4y = 0$$

$\rightarrow$ $y_1 = \cos 2x$ $\quad y_2 = \sin 2x$ $\rightarrow$ these live in the null space

note $y = 0$ also satisfies the differential eq.

$$(y_1 + y_2)'' + 4(y_1 + y_2) = 0$$

$$\underbrace{y_1'' + 4y_1}_{0} + \underbrace{y_2'' + 4y_2}_{0} = 0$$ closed under addition

$$(cy_1)'' + 4(cy_1) = 0$$

$$c\,y_1'' + 4c\,y_1 = 0$$

$$c(\underbrace{y_1'' + 4y_1}_{0}) = 0$$

closed under scalar multiplication

the null space of a linear transformation is also called the <u>kernel</u>

For an $m \times n$ matrix A the column space is the linear combination of columns of A

if $A = [\,\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_n\,]$

$$\text{Col } A = \text{span}\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$$

$$= \{\vec{b} : \vec{b} = A\vec{x} \text{ for some } \vec{x} \text{ in } \mathbb{R}^n\}$$

Nul A: associated w/ the domain of transformation

Col A: " " range " "

We can often recover A if we have the output vector

<u>example</u> $\left\{ \begin{bmatrix} -3r+2s+3t \\ -r-2s \\ r+3s-2t \\ 2r-3s+t \end{bmatrix} \right\}$ r, s, t are constants

$$r \begin{bmatrix} -3 \\ -1 \\ 1 \\ 2 \end{bmatrix} + s \begin{bmatrix} 2 \\ -2 \\ 3 \\ -3 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -3 & 2 & 3 \\ -1 & -2 & 0 \\ 1 & 3 & -2 \\ 2 & -3 & 1 \end{bmatrix}}_{A} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

Is Col A a subspace?

from 4.1, we know if $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ span some space and $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ are in a vector space, then span $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ is a subspace.

~~A linear transformation~~.

If we know $A\vec{x} = \vec{v}$ and $A\vec{x} = \vec{w}$ are both consistent, what can we say about ~~Bet~~ $A\vec{x} = \vec{v} + \vec{w}$?

if $A\vec{x} = \vec{v}$ is consistent, then $\vec{v}$ is in Col A

" $A\vec{x} = \vec{w}$ " " " $\vec{w}$ is " "

Col A is subspace so $\vec{v} + \vec{w}$ is also in Col A

therefore $\vec{v} + \vec{w}$ MUST be linear combo of columns of A

$\Rightarrow A\vec{x} = \vec{v} + \vec{w}$ is consistent

A Linear transformation maps a vector space V to another vector space W such that for each $\vec{x}$ in V

$$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$$

$$T(c\vec{u}) = cT(\vec{u})$$

implies $T(\vec{0}) = \vec{0}$

<u>example</u> $T : \mathbb{P}_2 \to \mathbb{R}^2$ by $T(\vec{p}) = \begin{bmatrix} \vec{p}(0) \\ \vec{p}(1) \end{bmatrix}$

2nd-deg polynomials

for example, if $\vec{p}(t) = 3 + 5t + 7t^2$

then $T(\vec{p}) = \begin{bmatrix} 3 \\ 15 \end{bmatrix}$

Is T linear transformation?

Find $\vec{p}$ in $\mathbb{P}_2$ that spans the kernel of T.

If T is linear, then $T(\vec{p}+\vec{q}) = T(\vec{p}) + T(\vec{q})$

and $T(c\vec{p}) = c\,T(\vec{p})$

(true, may help if think in terms of deriv. / integral or common operations w/ polynomials)

Find $\vec{p}$ in $\mathbb{P}_2$ that spans $\underbrace{\text{kernel of } T}$

$$T(\vec{p}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \vec{p}(0) \\ \vec{p}(1) \end{bmatrix}$$

this means $\vec{p}(t)$ is a multiple of $(t)(t-1)$

so $\boxed{\vec{p}(t) = c\,t\,(t-1)}$ this polynomial ("vector") spans the kernel

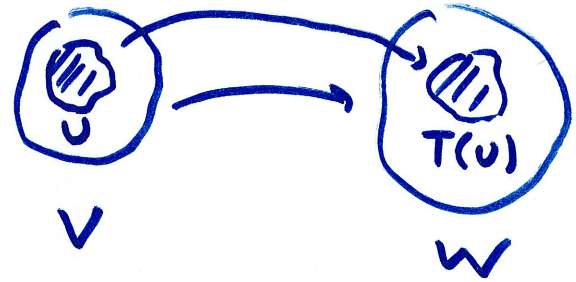
$$T(\vec{p}+\vec{q}) = \begin{bmatrix} (\vec{p}+\vec{q})(0) \\ (\vec{p}+\vec{q})(1) \end{bmatrix} = \begin{bmatrix} \vec{p}(0)+\vec{q}(0) \\ \vec{p}(1)+\vec{q}(1) \end{bmatrix} = \begin{bmatrix} \vec{p}(0) \\ \vec{p}(1) \end{bmatrix} + \begin{bmatrix} \vec{q}(0) \\ \vec{q}(1) \end{bmatrix}$$

$$T(c\vec{p}) = \begin{bmatrix} c\,\vec{p}(0) \\ c\,\vec{p}(1) \end{bmatrix} = c\begin{bmatrix} \vec{p}(0) \\ \vec{p}(1) \end{bmatrix}$$

<u>example</u> $T: V \to W$ $\qquad$ V, W are vector spaces

U is a subrpace of V

Is $T(U)$ a subrpace of W?



Since V is a vector space and U is a subrpace of V

$\vec{0}_V$ is in U, because T is linear, so

$T(\vec{0}_V) = \vec{0}_W$, which is in $T(U)$, so $T(U)$

contains zero vector.

Let $\vec{x}$ and $\vec{y}$ be vectors in U. Then $T(\underbrace{\vec{x}+\vec{y}}_{\text{in } U})$

$= T(\vec{x}) + T(\vec{y})$ because T is linear.

so $T(U)$ is closed under addition

$c\vec{u}$ is in U because U is a subspace and

T is linear so $T(c\vec{u}) = c\,T(\vec{u})$, this

shows $T(U)$ is closed under scalar multiplication.