5.3 Diagonalization (part 2)

example

Diagonalize $A = \begin{bmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

triangular, so eigenvalues are on the main diagonal

$\lambda = 5, 3, 2, 2$

find eigenvector for $\lambda = 5$

solve $(A - \lambda I)\vec{x} = \vec{0}$

$$\begin{bmatrix} 0 & -3 & 0 & 9 & 0 \\ 0 & -2 & 1 & -2 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \end{bmatrix}$$

$x_2 = x_3 = x_4 = 0$

x_1 free, choose $x_1 = 1$

so eigenvector $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\lambda = 5$

$\lambda = 3$

$$\begin{bmatrix} 2 & -3 & 0 & 9 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$x_3 = x_4 = 0$

x_2 free

$x_1 = 3/2 \, x_2$

choose $x_2 = 2$

$\vec{v} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ $\lambda = 3$

$\lambda = 2$

$$\begin{bmatrix} 3 & -3 & 0 & 9 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3, x_4 free

$x_2 = -x_3 + 2x_4$

$x_1 = x_2 - 3x_4$

$= -x_3 + 2x_4 - 3x_4 = -x_3 - x_4$

$$\vec{x} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

eigenspace has dimension of 2, $\lambda = 2$ has algebraic multiplicity of 2.

$A = PDP^{-1}$

$$P = \begin{bmatrix} 1 & 3 & -1 & -1 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$\hookrightarrow$ this factorization is NOT unique

because we can order columns of P however we want, as long as the corresponding eigenvalues are in the same columns of D. Also, we can scale columns of P however we want.

A matrix is NOT diagonalizable if at least one λ is repeated AND that λ's eigenspace does not have enough dimensions.

Simple example of non diagonalizable matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad \lambda = 1, 1$$

eigenvector for $\lambda = 1$: $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ x_2 is free

$(A - \lambda I)\vec{x} = \vec{0}$ $\qquad x_1 = 0$

eigenspace dimension $= 1$

$\lambda = 1$ has multiplicity of 2

the only eigenvector is $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

If A is diagonalizable, it may or may not be invertible, because if at least one $\lambda = 0$, then A^{-1} does not exist.

What about the other way around? Does invertibility imply diagonalizability? No, see $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

If A is diagonalizable and invertible, is A^{-1} diagonalizable?

if A is diagonalizable, then $A = PDP^{-1}$ diagonal matrix

if A is invertible, then D is invertible

$$A = PDP^{-1}$$

$$A^{-1} = (PDP^{-1})^{-1} \qquad \text{recall } (AB)^{-1} = B^{-1}A^{-1}$$

$$= (P^{-1})^{-1} D^{-1} P^{-1}$$

$$= P D^{-1} P^{-1}$$

is D^{-1} diagonal?

if $D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

then
$$\left[\begin{array}{ccc|ccc} a & 0 & 0 & 1 & 0 & 0 \\ 0 & b & 0 & 0 & 1 & 0 \\ 0 & 0 & c & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/a & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/b & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/c \end{array}\right]$$
where the right block is D^{-1}.

so D^{-1} is diagonal, and $A^{-1} = P D^{-1} P^{-1}$

so, A^{-1} is diagonalizable.

If A is 5×5 with two different eigenvalues. One eigenspace is 3-dimensional and the other eigenspace is 2-dimensional. Is A diagonalizable?

$$\lambda = \underbrace{\underbrace{\underbrace{a, a}_{\text{eigenspace is 2-D}}}_{\text{two eigenvectors}}, \underbrace{\underbrace{b, b, b}_{\text{eigenspace is 3-D}}}_{\text{three eigenvectors}}}_{\text{all linearly indp.}}$$

all linearly indp. So, yes, P matrix exists

If A is 4×4 w/ 3 ~~etg~~ different eigenvalues, one eigenspace is 1-dimensional and one of the other is two-dimensional. Is it possible that A is NOT diagonalizable?

$$\lambda = a, b, \underbrace{c, c}_{\text{eigenspace is 2-D}}$$

2 vectors

eigenspace 1-D

1 vector

b is an eigenvalue, so it has an eigenvector it must have eigenspace ~~of~~ that is 1-D and it is linearly indep from others because b is distinct from a and c

NO, A is always diagonalizable