MA 265 Final Exam

- Thursday, 8/2
- 3:30pm-5:30pm
- FRNY G140

- 20 multiple choice problems
- Covers all lessons
  - no special emphasis on material since exam 2

Fall 2012

4. T/F

i) $A\vec{x} = \vec{b}$ has m equations and n unknowns, $m < n$
then system has infinitely many solutions

FALSE because sys. could be inconsistent

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 0 \end{bmatrix}$$

ii) A, B are $n \times n$ and AB is nonsingular, then
A and B are nonsingular. TRUE

$(AB)^{-1}$ exists $\quad \underset{\uparrow \text{ exists}}{(AB)^{-1}} = \underbrace{B^{-1}A^{-1}}_{\text{must also exist}}$

iii) if A, B, C $n \times n$ such that $AB = AC$, then $B = C$.

FALSE. Because $A = 0$ means $AB = AC$ for any B, C

F12

5. i) If λ is an eigenvalue for A then $-\lambda$ is an eigenvalue for $-A$

$$A\vec{v} = \lambda\vec{v} \qquad -A\vec{v} = -\lambda\vec{v}$$

let $-A = B$ $\qquad B\vec{v} = \underbrace{(-\lambda)}_{\text{eigenvalue of } B=-A}\vec{v}$

TRUE.

ii) $\vec{v}$ is eigenvector for A then $\vec{v}$ is also e-vector for 2A

$$A\vec{v} = \lambda\vec{v} \qquad 2A\vec{v} = 2\lambda\vec{v} \qquad \text{let } 2A = C$$

$\hookrightarrow C\vec{v} = (2\lambda)\vec{v}$

$\hookrightarrow$ is e-vector for $C = 2A$

TRUE

iii) eigenspace for eigenvalue λ has dimension = multiplicity of λ

FALSE (but true for all symmetric matrices)

F12

8. $$\det \underbrace{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}}_{A} = 5$$

$$\det \begin{array}{c} \begin{array}{cccc} * & & & * \end{array} \\ \begin{bmatrix} a_4 & a_2 & a_3 & a_1 \\ b_4 & b_2 & b_3 & b_1 \\ 2c_4+3a_4 & 2c_2+3a_2 & 2c_3+3a_3 & 2c_1+3a_1 \\ d_4 & d_2 & d_3 & d_1 \end{bmatrix} \end{array} = ?$$

$\det(A^T) = \det(A)$ $\quad$ $*$: row swap within A^T

$\rightarrow$ det changes sign

det changed by the same { then ~~back~~ transpose again; mult. row 3 by 2

does not change det { then added 3 times row 1 to row 3

$$= 5 \cdot -1 \cdot 2 = -10$$

F12

11\. Dimension of subspace spanned by

$[1, 2, 3, 4, 5]$, $[0, 2, 0, 0, 0]$, $[0, 0, 0, 0, 3]$

$[2, 4, 6, 8, 13]$, $[2, 6, 6, 8, 10]$ is

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 2 \\ 2 & 2 & 0 & 4 & 6 \\ 3 & 0 & 0 & 6 & 6 \\ 4 & 0 & 0 & 8 & 8 \\ 5 & 0 & 3 & 13 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 3 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} \boxed{1} & 0 & 0 & 2 & 2 \\ 0 & \boxed{2} & 0 & 0 & 2 \\ 0 & 0 & \boxed{3} & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\dim = 3$

13. W spanned by $\left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

apply Gram-Schmidt to produce an orthonormal basis

2 vectors in basis because $\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\left\{ \underset{\vec{x}_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{\vec{x}_2}{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}} \right\}$ is a basis but not orthonormal

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\langle \vec{x}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \qquad \vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

19. A is 2×2 eigenvalues λ_1, λ_2 $\lambda_1 \neq \lambda_2$. which is/are true?

i) A is diagonalizable. TRUE, because the eigenvectors corresponding to <u>distinct</u> eigenvalues are ALWAYS linearly independant $\rightarrow P$ in $A = PDP^{-1}$

ii) $\{\vec{v_1}, \vec{v_2}\}$ is linearly indepent. TRUE, see above.

iii) $\{\vec{v_1}, \vec{v_2}\}$ is orthogonal. FALSE, in general.
(but ALWAYS true if $A = A^T$)

20. $L: \mathbb{R}^3 \to \mathbb{R}^2 \qquad L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ 3x_2 - 2x_3 \end{bmatrix}$

matrix for L ?

size of matrix ? 2×3

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \end{bmatrix}$$

$$\boxed{2 \times 3} \quad 3 \times 1 \qquad\qquad 2 \times 1$$

$$L\left(x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\left.\begin{aligned} L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) &= \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) &= \begin{bmatrix} 0 \\ -2 \end{bmatrix} \end{aligned}\right\} \to A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -2 \end{bmatrix}$$

22. $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ find least-squares solution

$$\boxed{A^T A \vec{x} = A^T \vec{b}}$$

$$A^T A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 6 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 14 & 7 & 2 \\ 7 & 6 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & -5 & 0 \\ 7 & 6 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 6/7 & 1/7 \\ 0 & 1 & 0 \end{bmatrix}$$

$x_2 = 0$

$x_1 = 1/7$

$$\hat{x} = \begin{bmatrix} 1/7 \\ 0 \end{bmatrix}$$

23. $A = \begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}$ Find P such that $P^{-1}AP = D$

D is diagonal.

$$P^{-1}AP = D \quad \longleftrightarrow \quad AP = PD$$

$$A = PDP^{-1}$$

$$\begin{vmatrix} 1-\lambda & 4 \\ 1 & -2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-2-\lambda) - 4 = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$(\lambda + 3)(\lambda - 2) = 0$$

$$\lambda = -3, \quad \lambda = 2 \qquad D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$$

$\lambda = 2$ $\begin{bmatrix} -1 & 4 & 0 \\ 1 & -4 & 0 \end{bmatrix}$

$$\vec{v}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

or $D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$

$\lambda = -3$ $\begin{bmatrix} 4 & 4 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad P = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}$$

25. $\vec{x}\,' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{x}$ $\qquad \vec{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $\quad \vec{x}(1) = ?$

need eigenthings

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 - 4 = 0$$

$$1-\lambda = \pm\sqrt{4} = \pm 2$$

$$\lambda_1 = -1 \qquad \lambda_2 = 3$$

$\lambda_1 = -1$ $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 3$ $\begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\vec{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 3 \\ -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix}$$

$c_2 = 2 \quad c_1 = 1$

$$\vec{x}(1) = \begin{bmatrix} e^{-1} \\ -e^{-1} \end{bmatrix} + \begin{bmatrix} 2e^3 \\ 2e^3 \end{bmatrix}$$