MA 26500 **Final Exam** **Summer 2018**

Name: ____________________________________

Instructor: ________________________________

Instructions:

1. This is a two-hour exam.
2. There are 20 problems on this exam. Each problem is worth 10 points.
3. No books, notes, or calculators are allowed.
4. Please turn off your cell phone.
5. Show relevant work for ALL problems. Problems with insufficient work may be marked as wrong, even if the answer is correct. There is no partial credit, but your work may be used to justify raising your borderline grade.
6. Circle one and only one choice for each problem.

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I agree to abide by the instructions above:

Signature: ________________________________

1. Let A be an $n \times n$ matrix and $\mathbf{b}$ be an $n \times 1$ matrix. Which of the following statements is/are true?

(i) $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if $\mathbf{b}$ can be expressed as a linear combination of the columns of the matrix A.

(ii) A is singular if and only if A can be expressed as a product of elementary matrices.

(iii) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if the reduced row echelon form of A is I_n.

(iv) $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if $\det(A) = 0$.

A. (ii) only

B. (iii) only

C. (i) and (iii) only

D. (ii) and (iv) only

E. (i), (ii) and (iii) only

2. Let A, B be an $n \times n$ invertible matrix, which of the following statements is/are **always** true?

(i) $(A+B)^{-1} = A^{-1} + B^{-1}$

(ii) $(A^3B^T)^{-1} = (B^{-1})^T(A^{-1})^3$

(iii) $\det(kA) = k\det(A)$ for any scalar k

(iv) $(kA)^{-1} = k^{-1}A^{-1}$ for any scalar $k \neq 0$

(v) $A + A^T$ is diagonalizable

A. (i) and (iii) only

B. (ii) and (iv) only

C. (iii), (iv) and (v) only

D. (ii), (iv) and (v) only

E. (i)-(v)

3. For a real number a, consider the following linear system,

$$\begin{cases} x + y + z = 2 \\ x + 2y + z = 3 \\ x + y + (a^2 - 3)z = a \end{cases}$$

Which of the following is true?

A. If $a = \sqrt{3}$ then the system has at least two distinct solutions.
B. If $a = 2$ then the system is inconsistent
C. If $a = \sqrt{3}$ then the system is inconsistent
D. If $a = -2$ then the system is inconsistent
E. If $a = \pm 2$ then the system has infinitely many solutions

4. Given that

$$\det \begin{bmatrix} a_1 & a_3 & \mathbf{a_2} & a_4 \\ b_1 + 2a_1 & b_3 + 2a_3 & \mathbf{b_2} + \mathbf{2a_2} & b_4 + 2a_4 \\ 3c_1 & 3c_3 & 3\mathbf{c_2} & 3c_4 \\ d_1 + b_1 & d_3 + b_3 & \mathbf{d_2} + \mathbf{b_2} & d_4 + b_4 \end{bmatrix} = 15$$

What is the determinant of

$$\begin{bmatrix} a_1 & \mathbf{a_2} & a_3 & a_4 \\ b_1 & \mathbf{b_2} & b_3 & b_4 \\ c_1 & \mathbf{c_2} & c_3 & c_4 \\ d_1 & \mathbf{d_2} & d_3 & d_4 \end{bmatrix}?$$

A. 5
B. -45
C. -5
D. -15
E. 90

5. Let

$$A = \begin{bmatrix} t & 0 & 0 & 1 \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 1 & 0 & 0 & t \end{bmatrix}$$

Determine all values of t so that A is nonsingular.

A. $t \neq \pm 2$

B. $t \neq 0$

C. $t \neq \pm 1$

D. $t = 0$ or $t = \pm 1$

E. $t \neq 0$ and $t \neq \pm 1$

6. Let $\mathcal{L} : \mathbb{P}_2 \to \mathbb{R}^2$ be the linear transformation satisfying

$$\mathcal{L}[1+t] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathcal{L}[t+t^2] = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathcal{L}[1+t^2] = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Then $\mathcal{L}[2t+4t^2]$ equals

A. $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

B. $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$

C. $\begin{bmatrix} 3 \\ 8 \end{bmatrix}$

D. $\begin{bmatrix} 1 \\ 6 \end{bmatrix}$

E. $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$

7. Given

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

What is the $(2, 3)$ entry of the inverse of A?

A. 0

B. 1

C. 1/2

D. -1

E. -2

8. For what values of a is the vector $\begin{bmatrix} a^2 \\ -3a \\ -2 \end{bmatrix}$ in

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\}$$

A. $a \neq 1$

B. $a \neq 2$

C. $a \neq 0$

D. $a = 1$ or $a = 2$

E. $a \neq 1$ and $a \neq 0$

9. Let A be a 7×4 matrix such that every solution of $A\mathbf{x} = \mathbf{0}$ belongs to

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 4 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \\ 3 \end{bmatrix} \right\}$$

What is the rank of A?

A. 1

B. 3

C. 4

D. 5

E. 7

10. What is the dimension of the column space of matrix A, where

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 1 & -5 & -2 & 1 \\ 1 & 7 & 17 & 10 & 3 \end{bmatrix}$$

A. 1

B. 2

C. 3

D. 4

E. 5

11. Which of the following statements are true?

(i) A nonhomogeneous system of 7 equations in 6 unknowns always has a unique solution.

(ii) If A and B are row equivalent then their row spaces are the same.

(iii) If a 7×9 matrix has 7 pivot columns then Nul A$=\mathbb{R}^2$

A. I only

B. II only

C. III only

D. II and III only

E. I, II, and III

12. Which of the following is an eigenvalue of

$$\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

A. -1

B. 0

C. 2

D. 4

E. 6

13. If $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$ has eigenvalues 2 and 1 and $A = PDP^{-1}$ where $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, find P.

A. $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 2 & 1 \\ -3 & -1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$

E. $\begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}$

14. The mapping $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by $T(a_0 + a_1 t + a_2 t^2) = a_1 + 2a_2 t$ is a linear transformation. What is the matrix of this linear transformation using the basis $\{1, t, t^2\}$?

A. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

C. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$

E. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

15. The system of differential equations $\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 2 & -4 \end{bmatrix} \mathbf{x}$ has a complex-valued solution $e^{(-1+i)t} \begin{bmatrix} 5 \\ 3-i \end{bmatrix}$. What is the general solution?

A. $C_1 e^{-t} \begin{bmatrix} 5\cos(t) \\ 3\cos(t) - \sin(t) \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} -5\sin(t) \\ -\cos(t) - 3\sin(t) \end{bmatrix}$

B. $C_1 e^{-t} \begin{bmatrix} 5\cos(t) \\ 3\cos(t) + \sin(t) \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 5\sin(t) \\ \cos(t) + 3\sin(t) \end{bmatrix}$

C. $C_1 e^{-t} \begin{bmatrix} 5\cos(t) \\ 3\cos(t) + \sin(t) \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 5\sin(t) \\ -\cos(t) + 3\sin(t) \end{bmatrix}$

D. $C_1 e^{-t} \begin{bmatrix} 5\cos(t) \\ 3\cos(t) - \sin(t) \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 5\sin(t) \\ -\cos(t) - 3\sin(t) \end{bmatrix}$

E. $C_1 e^{-t} \begin{bmatrix} 5\cos(t) \\ 3\cos(t) + \sin(t) \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} -5\sin(t) \\ -\cos(t) + 3\sin(t) \end{bmatrix}$

16. Let $\vec{y} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}$, $\vec{u_1} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$, $\vec{u_2} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $W = \text{span}\{\vec{u_1}, \vec{u_2}\}$. If the point in W that is closest to $\vec{y}$ is (a, b, c), find c.

A. -8

B. -1

C. 0

D. 2

E. 4

17. After using the Gram-Schmidt process, a basis for the column space of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ is found to be $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{2}{3} \\ b_3 \\ b_4 \end{bmatrix} \right\}$. What is b_4?

A. 0

B. $-\frac{1}{4}$

C. $-\frac{2}{3}$

D. $\frac{1}{3}$

E. 1

18. Find a least-squares solution of $A\vec{x} = \vec{b}$ where $A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}, \vec{b} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

A. $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

B. $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

C. $\begin{bmatrix} -4 \\ 11 \end{bmatrix}$

D. $\begin{bmatrix} 6 \\ -11 \end{bmatrix}$

E. $\begin{bmatrix} -11 \\ 22 \end{bmatrix}$

19. Which of the following statements are true?

(i) If $A^T = A$ and if $A\vec{u} = 3\vec{u}$ and $A\vec{v} = 4\vec{v}$, then $\vec{u}^T\vec{v} = 0$

(ii) An orthogonal matrix is orthogonally diagonalizable

(iii) The dimension of an eigenspace of a symmetric matrix is sometimes less than the multiplicity of the corresponding eigenvalue

A. I only

B. I and II only

C. II and III only

D. All of them are true

E. None of them is true

20. Let V be the space $C[-1,1]$ (vector space of all continuous functions on $-1 \leq t \leq 1$) with the inner product defined as $< f, g >= \int_{-1}^{1} f(t)g(t)\, dt$. Find an orthogonal basis for the subspace spanned by the polynomials 1, t, and t^2.

A. $\{1, 2t-1, 12t^2\}$

B. $\{1, 2t-1, 3t^2-2\}$

C. $\{1, t, 3t^2-2\}$

D. $\{1, 2t-1, 12t^2-12t+2\}$

E. $\{1, t, 3t^2-1\}$