Nonlinear system

critical points / equilibrium

linearize about each cp, determine type and stability

$$x' = f(x,y) \qquad y' = g(x,y)$$

Jacobian $J(x_0,y_0) = \begin{bmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \\ \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \end{bmatrix}$

$$x' = -y - x^2$$
$$y' = -x + y^2$$

cp: $-y - x^2 = 0 \rightarrow y = -x^2$

$-x + y^2 = 0$

$\hookrightarrow -x + (-x^2)^2 = 0$

$x^4 - x = 0$

$x(x^3 - 1) = 0$ $\qquad x = 0, \quad x = 1$

$\qquad\qquad\qquad\qquad y = 0, \quad y = -1$

$(0,0), (1,-1)$

$$J = \begin{bmatrix} -2x & -1 \\ -1 & 2y \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad \cancel{x=}$$

$$\begin{vmatrix} -\lambda & -1 \\ -1 & -\lambda \end{vmatrix} = 0 \quad \lambda^2 - 1 = 0 \quad \lambda = 1, -1$$

saddle

$$J(1,-1) = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$$

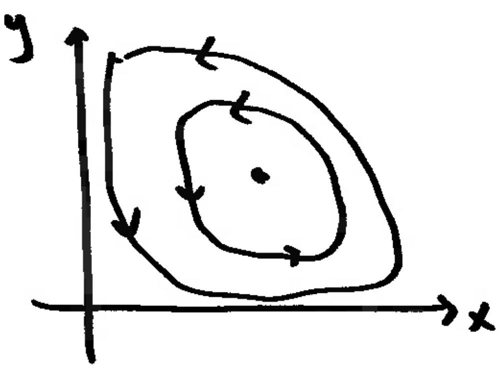
$$\begin{vmatrix} -2-\lambda & -1 \\ -1 & -2-\lambda \end{vmatrix} = 0$$

$$(-2-\lambda)^2 - 1 = 0$$

$$-2-\lambda = \pm 1 \quad \lambda = -3, \quad \lambda = -1$$

sink

Predator-Prey



asymp. stable : $t \to \infty$ $\vec{x} \to$ some point (eigenvalues w/ neg. real part)

stable : orbits a point

IVP w/ unit step

$$y'' + 4y = f(t) \qquad f(t) = \begin{cases} 1 & 0 \le t < 3 \\ t & 3 \le t < \infty \end{cases}$$

$$y(0) = y'(0) = 0$$

write $f(t)$ in terms of unit step $\quad u(t-c) = \begin{cases} 1 & t \ge c \\ 0 & \text{else} \end{cases}$

reset to zero

$$f(t) = 1 + \underbrace{u(t-3)}_{\text{activate something at } t=3}(-1 + t)$$

want this

LT on both sides

$$s^2 Y - \cancel{sy(0)}^{0} - \cancel{y'(0)}^{0} + 4Y = \mathcal{L}\{1\} + \mathcal{L}\{u(t-3)(-1+t)\}$$

$$(s^2+4)Y = \frac{1}{s} + e^{-3s}\mathcal{L}\{-1+t+3\}$$

shift LEFT in t by the delay from step

$t \to t+c$

$$= \frac{1}{s} + e^{-3s}\mathcal{L}\{t+2\}$$

$$= \frac{1}{s} + e^{-3s}\left(\frac{1}{s^2} + \frac{2}{s}\right)$$

$$Y = \frac{1}{s(s^2+4)} + e^{-3s}\frac{1}{s^2(s^2+4)} + e^{-3s}\frac{2}{s(s^2+4)}$$

inv. LT

$$y = \mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4}\right\}$$

$$+ \mathcal{L}^{-1}\Big\{e^{-3s}\cdot\underbrace{\left(\frac{1}{4s^2} - \frac{1}{4}\frac{1}{s^2+4}\right)}_{\frac{1}{4}t - \frac{1}{8}\sin 2t}\Big\}$$

$$+ \mathcal{L}^{-1}\left\{e^{-3s}\left(\frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s}{s^2+4}\right)\right\}$$

$$\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$

$$1 = As^2 + 4A + Bs^2 + Cs$$

$$1 = (A+B)s^2 + Cs + 4A$$

$A+B=0$ $\quad B = -\frac{1}{4}$

$C = 0$

$4A = 1 \to A = \frac{1}{4}$

$$\frac{1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$$

$$= \frac{1}{4}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s^2+4}$$

$$= \frac{1}{4} - \frac{1}{4}\cos 2t + u(t-3)\left(\frac{1}{4}(t-3) - \frac{1}{8}\sin 2(t-3)\right)$$

$$+ u(t-3)\left(\frac{1}{2} - \frac{1}{2}\cos 2(t-3)\right)$$

$$f(t) * g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

$$\mathcal{L}\{f * g\} = F(s)G(s)$$

$$\frac{1}{s(s^2+4)} = \underbrace{\frac{1}{s}}_{F} \cdot \underbrace{\frac{1}{s^2+4}}_{G}$$

$f(t) = 1$

$g(t) = \frac{1}{2}\sin 2t$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \int_0^t (1)\cdot \frac{1}{2}\sin 2\tau \, d\tau = -\frac{1}{4}\cos 2\tau \Big|_0^t$$

$$= -\frac{1}{4}\cos 2t + \frac{1}{4}$$