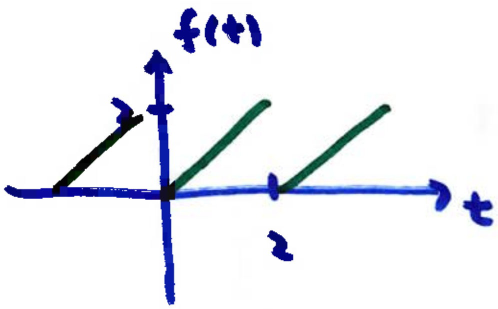
9.2 #17

$f(t) = t \quad 0 < t < 2 \quad$ period 2

show that $f(t) = 1 - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\pi t}{n}$

show that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \frac{\pi}{4}$



$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{n\pi t}{L}\right) dt \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n\pi t}{L}\right) dt$

L is half period, so $L = 1$

$a_n = \frac{1}{L} \int_{0}^{2L} f(t) \cos\left(\frac{n\pi t}{L}\right) dt$

$$a_0 = \int_0^2 t\,dt = \frac{1}{2}\cdot 2\cdot 2 = 2$$

$$a_n = \int_0^2 t\cos\left(\frac{n\pi t}{1}\right)dt = \frac{\cos(n\pi t)}{(n\pi)^2} - \frac{t\sin(n\pi t)}{n\pi}\Bigg|_0^2 = 0$$

$$b_n = \int_0^2 t\sin(n\pi t)\,dt = \dots$$

$$t \sim 1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi t)}{n}$$

Fourier series converges to $f(t)$ wherever $f(t)$ is continuous to $\frac{f(t^-)+f(t^+)}{2}$ (here, 1) if $f(t)$ is discontinuous

$$= 1 - \frac{2}{\pi}\frac{\sin(\pi t)}{1} - \frac{2}{\pi}\frac{\sin(2\pi t)}{2} - \frac{2}{\pi}\frac{\sin(3\pi t)}{3} - \frac{2}{\pi}\frac{\sin(4\pi t)}{4}) - \dots$$

$$= 1 - \frac{2}{\pi}\left(\frac{\sin(\pi t)}{1} + \frac{\sin(2\pi t)}{2} + \frac{\sin(3\pi t)}{3} + \frac{\sin(4\pi t)}{4} + \dots\right)$$

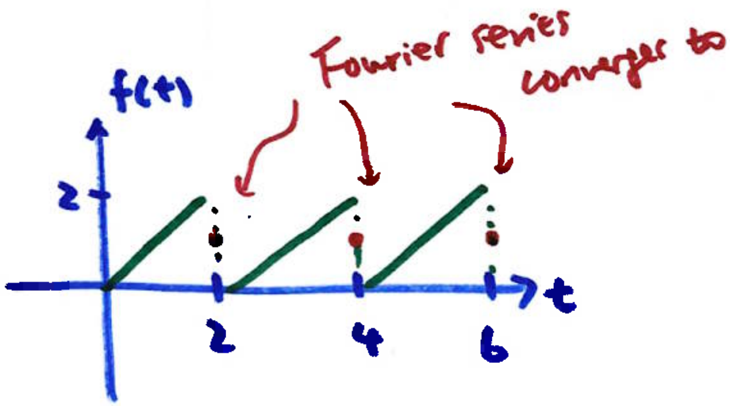
choose t so ones w/ even denom $= 0$

if $t=\frac{1}{2}$, $\sin(n\pi t)=0$ if n is even

$f(t)=t$ is continuous, so converges to $f(\frac{1}{2})=\frac{1}{2}$

$$\frac{1}{2} = 1-\frac{2}{\pi}\Big(\underbrace{1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots}_{\text{Leibniz series}}\Big)$$

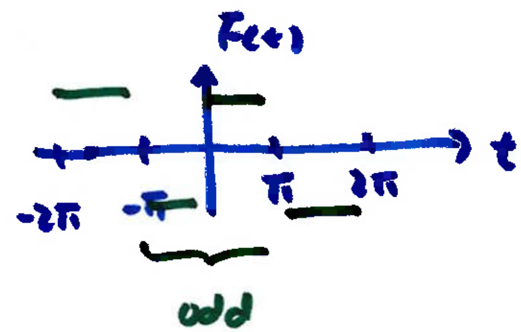
$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots = \frac{\pi}{4}$$



9.4 #1 $x'' + 5x = F(t)$ $F(t) = \begin{cases} 3 & 0 < t < \pi \\ -3 & \pi < t < 2\pi \end{cases}$ period 2π

Steady-periodic solution

$F(t)$ as a Fourier series



Sines series $a_n = 0$

$$b_n = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

$$= \frac{2}{\pi}\int_0^{\pi} 3\sin(nt)\,dt$$

$$= \frac{6}{\pi}\left(-\frac{\cos(nt)}{n}\right)\Big|_0^{\pi}$$

$$= -\frac{6}{\pi}\left(\frac{\cos(n\pi)-1}{n}\right)$$

$$b_n = \frac{6}{n\pi}\left(1-(-1)^n\right)$$

$F(t)$ is sine series $F(t) = \sum_{n=1}^{\infty} \frac{6(1-(-1)^n)}{n\pi} \sin(nt)$

express steady-state solution x_p as sine series, too

$$x_p = \sum_{n=1}^{\infty} B_n \sin(nt)$$ plug into $x'' + 5x = F(t)$

$$x_p' = \sum_{n=1}^{\infty} n B_n \cos(nt)$$

$$x_p'' = \sum_{n=1}^{\infty} -n^2 B_n \sin(nt)$$

$$-\sum_{n=1}^{\infty} n^2 B_n \sin(nt) + 5 \sum_{n=1}^{\infty} B_n \sin(nt) = \sum_{n=1}^{\infty} \frac{6(1-(-1)^n)}{n\pi} \sin(nt)$$

for each n,

$$-n^2 B_n + 5 B_n = \frac{6(1-(-1)^n)}{n\pi}$$

$$B_n = \frac{6(1-(-1)^n)}{n\pi(5-n^2)} \quad n = 1, 2, 3, \ldots$$

Potentially Useless Information

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right) dt$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right) dt$$

$$\int t\sin(at)dt = \frac{\sin(at)}{a^2} - \frac{t\cos(at)}{a} + C$$

$$\int t\cos(at)dt = \frac{\cos(at)}{a^2} + \frac{t\sin(at)}{a} + C$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(2\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$$

$$\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$$

$$\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$$

$$\cos(a)\sin(b) = \frac{1}{2}\left[\sin(b+a) + \sin(b-a)\right]$$

$$\int_{-L}^{L} \cos\left(\frac{n\pi t}{L}\right)\cos\left(\frac{m\pi t}{L}\right) dt = \begin{cases} 2L & \text{if } n = m = 0 \\ L & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi t}{L}\right)\sin\left(\frac{m\pi t}{L}\right) dt = \begin{cases} 0 & \text{if } n = m = 0 \\ L & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi t}{L}\right)\cos\left(\frac{m\pi t}{L}\right) dt = 0$$

Function	Transform	Function	Transform
$f(t)$	$F(s)$	e^{at}	$\dfrac{1}{s-a}$
$af(t)+bg(t)$	$aF(s)+bG(s)$	$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$f'(t)$	$sF(s)-f(0)$	$\cos kt$	$\dfrac{s}{s^2+k^2}$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$	$\sin kt$	$\dfrac{k}{s^2+k^2}$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	$\cosh kt$	$\dfrac{s}{s^2-k^2}$
$\int_0^t f(\tau)\,d\tau$	$\dfrac{F(s)}{s}$	$\sinh kt$	$\dfrac{k}{s^2-k^2}$
$e^{at}f(t)$	$F(s-a)$	$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$	$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	$\dfrac{1}{2k^3}(\sin kt - kt\cos kt)$	$\dfrac{1}{(s^2+k^2)^2}$
$tf(t)$	$-F'(s)$	$\dfrac{t}{2k}\sin kt$	$\dfrac{s}{(s^2+k^2)^2}$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$\dfrac{1}{2k}(\sin kt + kt\cos kt)$	$\dfrac{s^2}{(s^2+k^2)^2}$
$\dfrac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	$u(t-a)$	$\dfrac{e^{-as}}{s}$
$f(t)$, period p	$\dfrac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	$\delta(t-a)$	e^{-as}
1	$\dfrac{1}{s}$	$(-1)^{[\![t/a]\!]}$ (square wave)	$\dfrac{1}{s}\tanh\dfrac{as}{2}$
t	$\dfrac{1}{s^2}$	$\left[\!\left[\dfrac{t}{a}\right]\!\right]$ (staircase)	$\dfrac{e^{-as}}{s(1-e^{-as})}$
t^n	$\dfrac{n!}{s^{n+1}}$		
$\dfrac{1}{\sqrt{\pi t}}$	$\dfrac{1}{\sqrt{s}}$		
t^a	$\dfrac{\Gamma(a+1)}{s^{a+1}}$		