Exam 3 Review

Separation of variables

$$x u_{xx} + u_t = 0$$

assume $u(x,t) = X(x)\,T(t)$

$$u_{xx} = X''T \qquad u_t = XT'$$

$$xX''T + XT' = 0$$

$$xX''T = -XT'$$

$$\frac{xX''}{X} = -\frac{T'}{T} = c$$

ODEs: $\dfrac{xX''}{X} = c \rightarrow xX'' - cX = 0$

$$-\frac{T'}{T} = c \rightarrow T' + cT = 0$$

$$u_{xx} + u_{xt} + u_t = 0 \qquad u = XT$$

$$X''T + X'T' + XT' = 0$$

$$X''T = -X'T' - XT'$$

$$X''T = -T'(X' + X)$$

$$\frac{X''}{X' + X} = -\frac{T'}{T} = c$$

$$X'' - c(X' + X) = 0$$

$$T' + cT = 0$$

Heat Equation

$ku_{xx} = u_t \qquad 0<x<L$

BC's: $u(0,t) = u(L,t) = 0$ ends at zero

$u_x(0,t) = u_x(L,t) = 0$ ends insulated

or combinations

$u(0,t) = u_x(L,t) = 0$ right insulated ~~at 0~~, ~~right~~ left frozen

IC: $u(x,0) = f(x)$

eigenfunction for X depends on the BC's

we've worked out the first two

let's look at $u(0,t) = u_x(L,t) = 0$

$u = XT$

$ku_{xx} = u_t \rightarrow kX''T = XT'$

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda$$

$$\frac{X''}{X} = -\lambda \rightarrow X'' + \lambda X = 0$$

$u(0,t)=0 \rightarrow X(0)T(t)=0 \rightarrow X(0)=0$

$u_x(L,t)=0 \rightarrow X'(L)T(t)=0 \rightarrow X'(L)=0$

$$X = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$$

$$X(0) = 0 \rightarrow 0 = c_1$$

$$X' = c_2 \sqrt{\lambda} \cos(\sqrt{\lambda}\, x)$$

$$X'(L) = 0 \rightarrow 0 = c_2 \sqrt{\lambda} \cos(\sqrt{\lambda}\, L) \qquad c_2 \neq 0$$

$$\cos(\sqrt{\lambda}\, L) = 0$$

cosine $= 0$ at odd multiples of $\frac{\pi}{2}$

$$\sqrt{\lambda}\, L = \frac{2n-1}{2}\pi \qquad n = 1, 2, 3, \ldots$$

$$\sqrt{\lambda} = \frac{(2n-1)\pi}{2L}$$

$$\lambda = \frac{(2n-1)^2 \pi^2}{4L^2}$$

eigenfunction: $X = \sin(\sqrt{\lambda}x) = \sin\left(\frac{(2n-1)\pi}{2L}x\right)$

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switch BC: $u_x(0,t) = u(L,t) = 0$

$X'(0) = 0 \qquad X(L) = 0$

$X = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$

$X' = -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + C_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$

$X'(0) = 0 \rightarrow 0 = C_2\sqrt{\lambda} \rightarrow C_2 = 0$

so $X_n = \cos(\sqrt{\lambda}x)$

Heat Eq. (nonhomogeneous BC's)

$ku_{xx} = u_t$

$u(0,t) = T_1 \qquad u(L,t) = T_2$

Steady-state: $u_t = 0 \rightarrow u_{xx} = 0$ so $u(x) = T_1 + \frac{T_2 - T_1}{L}x$

Wave Eq.

$a^2 y_{xx} = y_{tt}$ $\quad 0 < x < L$

BC's: $y(0,t) = y(L,t) = 0$ ends fixed

$y_x(0,t) = y_x(L,t) = 0$ ends can move up/down

or some mix

space solution is the same as in heat eq.

IC's: $y(x,0) = f(x)$ initial displacement

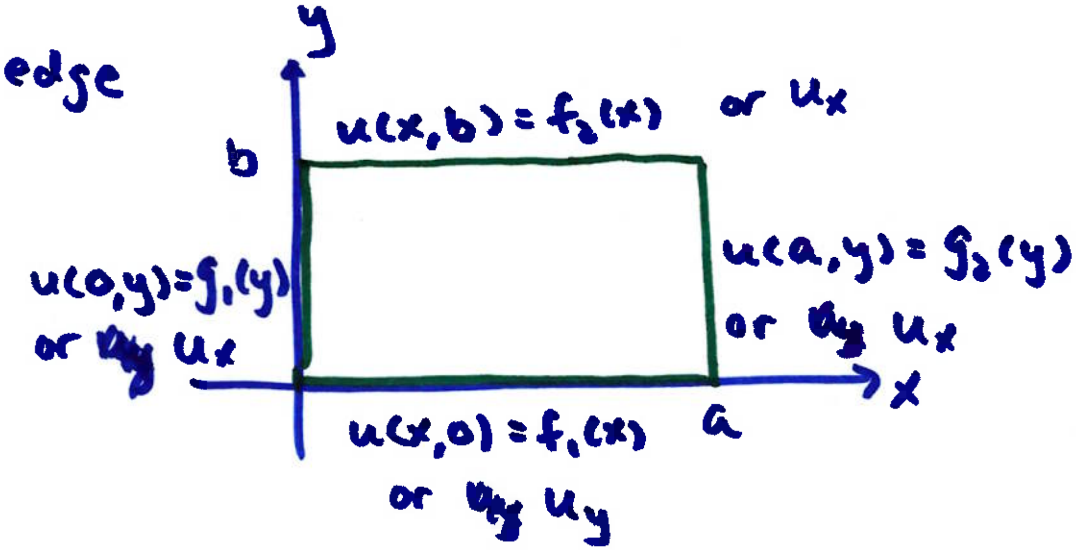
$y_t(x,0) = g(x)$ " velocity

T is from 2nd-order eq $\rightarrow$ cosine and sine

Laplace's Eq

$u_{xx} + u_{yy} = 0 \qquad 0 < x < a, \qquad 0 < y < b$

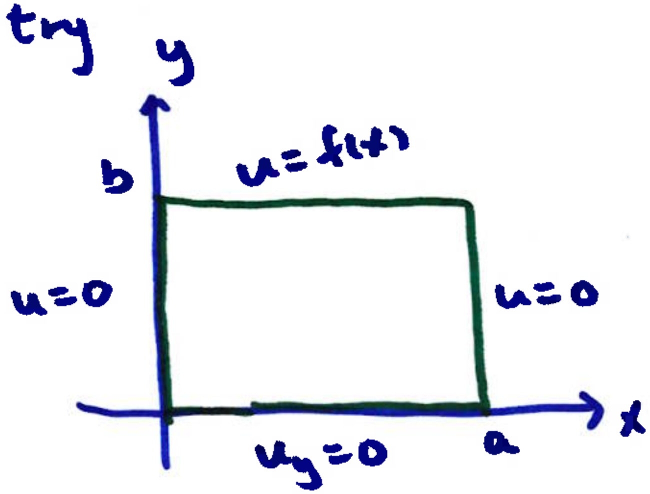
BC's on each edge



each can be u or partial of u

usually, 3 sides set to 0 (homogeneous)

Let's try



$$\left.\begin{aligned} u(0,y)=0 &\rightarrow X(0)=0 \\ u(a,y)=0 &\rightarrow X(a)=0 \end{aligned}\right] \text{ solve } \cancel{\text{this}}\ X \text{ first (two BC's)}$$

$u_y(x,0)=0 \rightarrow Y'(0)=0$

$u(x,b) = f(x)$

$u_{xx} + u_{yy} = 0$

$$X''Y + XY'' = 0$$

$$X''Y = -XY''$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$X(0) = X(a) = 0$

$X'' + \lambda X = 0$

$$X = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$$

$$0 = c_1$$

$$0 = c_2 \sin(\sqrt{\lambda} a)$$

$$\sin(\sqrt{\lambda} a) = 0 \qquad \sqrt{\lambda} a = n\pi \qquad n = 1, 2, 3, \ldots$$

$$\lambda = \frac{n^2 \pi^2}{a^2}$$

$$X = \sin\left(\frac{n\pi x}{a}\right)$$