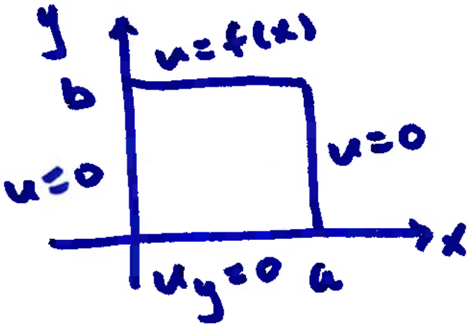
Finish the Laplace's eq. problem from last time

$u_{xx} + u_{yy} = 0$



$u = XY$

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$X'' + \lambda X = 0 \qquad X(0) = X(a) = 0$

$\vdots$

$\lambda_n = \dfrac{n^2\pi^2}{a^2} \qquad X_n = \sin\left(\dfrac{n\pi x}{a}\right)$

now $Y'' - \lambda Y = 0 \qquad Y'(0) = 0$ (bottom edge)

$$Y'' - \frac{n^2\pi^2}{a^2}Y = 0 \qquad Y = c_1 e^{\frac{n\pi}{a}y} + c_2 e^{-\frac{n\pi}{a}y}$$

or

$$Y = c_1 \cosh\left(\frac{n\pi y}{a}\right) + c_2 \sinh\left(\frac{n\pi y}{a}\right)$$

$$Y' = c_1 \cdot \frac{n\pi}{a} \sinh\left(\frac{n\pi y}{a}\right) + c_2 \cdot \frac{n\pi}{a} \cosh\left(\frac{n\pi y}{a}\right)$$

$$Y'(0) = 0 \rightarrow 0 = c_2 \cdot \frac{n\pi}{a} \rightarrow c_2 = 0$$

so, $Y_n = \cosh\left(\frac{n\pi y}{a}\right)$ $\quad u_n = \sin\left(\frac{n\pi x}{a}\right)\cosh\left(\frac{n\pi y}{a}\right) \quad n = 1, 2, 3, \ldots$

general solution

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right)\cosh\left(\frac{n\pi y}{a}\right)$$

$u(x,b) = f(x)$ (top edge)

$$f(x) = \sum_{n=1}^{\infty} \underbrace{\left[c_n \cosh\left(\frac{n\pi b}{a}\right)\right]}_{\text{constant of sine series}} \sin\left(\frac{n\pi x}{a}\right)$$

$$c_n \cosh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

Sturm-Liouville

$$y'' + \lambda y = 0 \qquad a < x < b$$

$$\alpha_1 y(a) - \alpha_2 y'(a) = 0$$

$$\beta_1 y(b) + \beta_2 y'(b) = 0$$

$\alpha_1, \alpha_2, \beta_1, \beta_2$ nonnegative $\rightarrow$ λ also nonnegative

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$$y'' + \lambda y = 0 \qquad 0 < x < L$$

$$h y(0) - y'(0) = 0 \qquad (h > 0)$$

$$y(L) = 0$$

check $\lambda < 0$

for convenience, let $\lambda = -k^2 \quad (k > 0)$

$$y'' - k^2 y = 0$$

$$y = C_1 \cosh(kx) + C_2 \sinh(kx)$$

$$y' = C_1 k \sinh(kx) + C_2 k \cosh(kx)$$

$hy(0) - y'(0) = 0$

$hC_1 - C_2 K = 0 \qquad C_1 = \frac{K}{h} C_2$

$y(L) = 0$

$C_1 \cosh(KL) + C_2 \sinh(KL) = 0$

$\frac{K}{h} C_2 \cosh(KL) + C_2 \sinh(KL) = 0$

$C_2 \Big(\underbrace{\frac{K}{h} \cosh(KL) + \sinh(KL)}_{=0} \Big) = 0 \qquad C_2 \neq 0$ (otherwise $C_1 = 0$)
$\rightarrow$ trivial

$$\frac{K}{h} \cosh(KL) + \sin(KL) = 0$$

$$\frac{1}{2}\frac{K}{h}\left(e^{KL} + e^{-KL}\right) + \frac{1}{2}\left(e^{KL} - e^{-KL}\right) = 0$$

$$\left(\frac{K}{h} + 1\right)e^{KL} + \left(\frac{K}{h} - 1\right)e^{-KL} = 0$$

solution only if $KL = 0 \rightarrow K = 0$

but we assumed $K > 0$ to start, so this cannot happen

so, no negative λ

now try $\lambda = 0$

$$y'' + \lambda y = 0$$

$$hy(0) - y'(0) = 0$$

$$y(L) = 0$$

if $\lambda = 0$, $y'' = 0$ so $y = c_1 x + c_2$

$$y' = c_1$$

$$hy(0) - y'(0) = 0$$

$$hc_2 - c_1 = 0 \rightarrow c_1 = hc_2$$

$$y(L) = 0$$

$$c_1 L + c_2 = 0 \rightarrow hc_2 L + c_2 = 0$$

$$c_2 (\underbrace{hL + 1}) = 0 \qquad c_2 \neq 0$$

$$hL = -1$$

can't happen since $h > 0$, $L > 0$

so no nontrivial y, so $\lambda \neq 0$

now $\lambda > 0$

$$y'' + \lambda y = 0$$

$$hy(0) - k'yy(0) = 0$$

$$y(L) = 0$$

let $\lambda = k^2 \quad (k > 0)$

$$y'' + k^2 y = 0 \rightarrow y = c_1 \cos(kx) + c_2 \sin(kx)$$

$$y' = -c_1 k \sin(kx) + c_2 k \cos(kx)$$

$$hy(0) - y'(0) = 0$$

$$hc_1 - c_2 k = 0 \qquad c_1 = \frac{k}{h} c_2$$

$$y(L) = 0$$

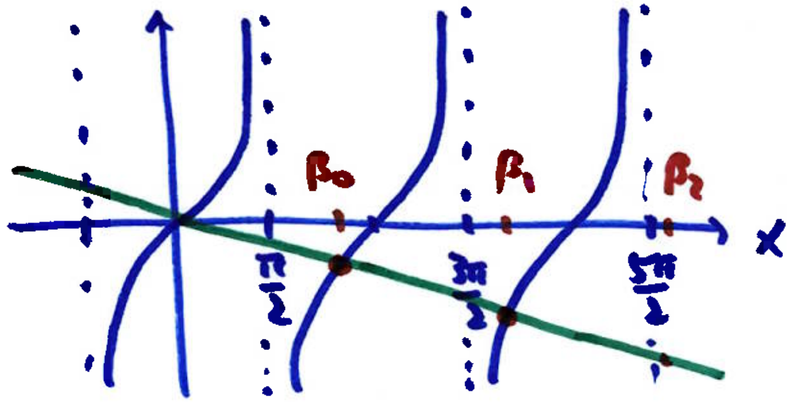
$$c_1 \cos(kL) + c_2 \sin(kL) = 0$$

$$\frac{k}{h} c_2 \cos(kL) + c_2 \sin(kL) = 0 \qquad c_2 \neq 0$$

$$\frac{k}{h} \cos(kL) + \sin(kL) = 0$$

$$\sin(KL) = -\frac{K}{h}\cos(KL)$$

$$\tan(KL) = -\frac{K}{h} = -\frac{KL}{hL} \rightarrow \text{solve } \tan(x) = -\frac{1}{hL}x$$



$\beta_0, \beta_1, \ldots$ are solutions

$$\beta_n = KL$$

$$\frac{\beta_n}{L} = K \quad \text{and} \quad \lambda = K^2$$

$$\lambda_n = \frac{\beta_n^2}{L^2} \text{ eigenvalues}$$

eigenfunctions: $y = c_1\cos(Kx) + c_2\sin(Kx)$

$$= c_1\cos\left(\frac{\beta_n}{L}x\right) + c_2\sin\left(\frac{\beta_n}{L}x\right) \qquad c_1 = \frac{K}{h}c_2$$

$$= c_2\underbrace{\left(\frac{K}{h}\cos\left(\frac{\beta_n}{L}x\right) + \sin\left(\frac{\beta_n}{L}x\right)\right)}_{\text{eigenfunctions}}$$

$$y_n = \frac{\beta_n}{hL}\cos\left(\frac{\beta_n}{L}x\right) + \sin\left(\frac{\beta_n}{L}x\right)$$