5.2 The Eigenvalue Method for Linear Systems

review linear systems $\vec{x}\,' = A\vec{x}$

in Ch. 6 we look at nonlinear systems

example system: $x_1' = x_1 + 2x_2$ goal: $x_1(t) = ?$

$x_2' = 2x_1 + x_2$ $x_2(t) = ?$

matrix form:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

form $\vec{x}\,' = A\vec{x}$

general solution:

$$\boxed{\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2}$$

$\lambda_1, \vec{v}_1$
$\lambda_2, \vec{v}_2$ are the eigenvalue/eigenvector pairs

eigenvalues: $\det(A-\lambda I)=0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 - 4 = 0$$

$(1-\lambda)^2 = 4$ $\qquad$ $1-\lambda = 2$ $\quad$ or $\quad$ $1-\lambda = -2$

$\lambda_1 = -1$ $\qquad$ $\lambda_2 = 3$

eigenvectors: Solve $(A-\lambda I)\vec{v} = \vec{0}$ using λ's we found

for $\lambda = -1$

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix}$$ augmented matrix

→ row reduction

$$\rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$$

b is free, so let $b = r$

row 1: $1 \cdot a + 1 \cdot b = 0$

$a = -b = -r$

$\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -r \\ r \end{bmatrix}$ choose $r \neq 0$

$r = 1$

$$\boxed{\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ eigenvector w/ eigenvalue } \lambda = -1}$$

for $\lambda_2 = 3$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

general solution:

$$\boxed{\vec{x} = c_1 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

particular solution: c_1, c_2 known

usually from initial conditions

$x_1(0) = 1$

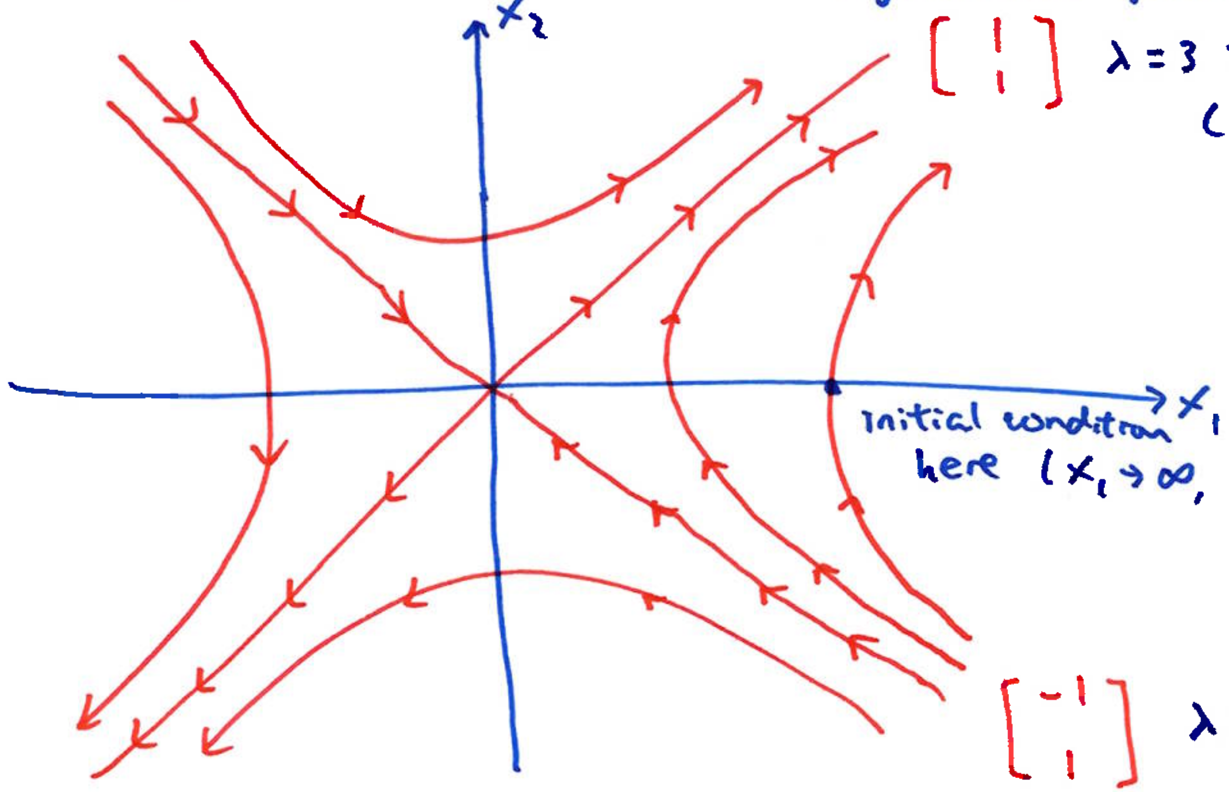
$x_2(0) = 3$ (for example)

graphical representation of $\vec{x}_\bullet = c_1 e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(phase diagram / phase portrait)

eigenvectors form asymptotes

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\lambda = 3 > 0$ (away from origin)



initial condition here ($x_1 \to \infty$, $x_2 \to \infty$)

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\lambda = -1 < 0$ (into origin)

Complex eigenvalues

$$\vec{x}\,' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}$$

eigenvalues: $\begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0$

$$(1-\lambda)^2 + 1 = 0$$

$$(1-\lambda)^2 = -1$$

$$1-\lambda = i \quad , \quad 1-\lambda = -i$$

$$\lambda_1 = 1-i, \quad \lambda_2 = 1+i$$

complex conjugate pairs

eigenvectors: for $\lambda_1 = 1-i$ $\lambda_2 = 1+i$

$$\vec{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

form <u>one</u> solution: $e^{\lambda_1 t}\vec{v}_1$ or $e^{\lambda_2 t}\vec{v}_2$

$$e^{(1-i)t}\begin{bmatrix} i \\ 1 \end{bmatrix} = e^{t}e^{-it}\begin{bmatrix} i \\ 1 \end{bmatrix}$$

Euler's identity

$$e^{it} = \cos t + i\sin t$$

$$e^{-it} = e^{i(-t)}$$

$$= \cos(-t) + i\sin(-t)$$

$$= \cos t - i\sin t$$

$$= e^{t}(\cos t - i\sin t)\begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$= e^{t}\begin{bmatrix} i\cos t - i^2\sin t \\ \cos t - i\sin t \end{bmatrix} = e^{t}\begin{bmatrix} \sin t + i\cos t \\ \cos t - i\sin t \end{bmatrix}$$

separate into real and imaginary parts.

$$= \underbrace{e^{t}\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}}_{\vec{u}} + i\,\underbrace{e^{t}\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}_{\vec{v}}$$

$\vec{u}$: real part

$\vec{v}$: imag part (no i)

general solution : $\vec{x} = c_1\vec{u} + c_2\vec{v}$

for this example, $\boxed{\vec{x} = c_1 e^{t}\begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + c_2 e^{t}\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}$

phase diagram is spirals

into origin : real part of λ is negative

out of origin : " " positive

for this example,