7.2 Laplace Transform of Initial Value Problems

$y'' + 3y' + 6y = t \qquad y(0) = 0, \; y'(0) = 1$

"old" way of solving this from MA 262/266

1. solve solution of complementary problem

$$y'' + 3y' + 6y = 0$$
$$\vdots$$
$$y = \ldots$$

2. nonhomogeneous part $\rightarrow f(t) = t$

~~no~~ undetermined coefficients

variation of parameters

3. use initial values to find values of constants

process is lengthy

today we'll use LT to solve this kind of problem

$$\mathcal{L}\{y\} = Y$$

$$\mathcal{L}\{y'\} = \int_0^\infty y' e^{-st}\,dt$$

$$= \lim_{a\to\infty} \int_0^a y' e^{-st}\,dt$$

by parts $\quad u = e^{-st} \quad dv = y'\,dt$

$du = -se^{-st}\,dt \quad v = y$

$$= \lim_{a\to\infty} \left(ye^{-st}\Big|_0^a + \int_0^a y\,se^{-st}\,dt \right)$$

$$= \lim_{a\to\infty} ye^{-st}\Big|_0^a + \lim_{a\to\infty} s \cdot \int_0^a ye^{-st}\,dt$$

$$= \lim_{a\to\infty} \Big(\underbrace{\cancel{ye^{-sa}}}_{s>0}{}^{\,0} - y(0) \Big) + s \cdot \underbrace{\lim_{a\to\infty} \int_0^a ye^{-st}\,dt}_{\mathcal{L}\{y\} = Y}$$

$$\boxed{\mathcal{L}\{y'\} = sY - y(0)}$$

likewise,

$$\boxed{\mathcal{L}\{y''\} = s^2 Y - sy(0) - y'(0)}$$

<u>example</u> $y'' + y = 4\cos 3t \qquad y(0)=1,\ y'(0)=0$

LT on both sides (table look up)

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = 4\mathcal{L}\{\cos 3t\}$$

$$s^2Y - sy(0) - y'(0) + Y = 4 \cdot \frac{s}{s^2+3^2}$$

$$(s^2+1)Y - s = \frac{4s}{s^2+3^2}$$

solve for Y

$$(s^2+1)Y = \frac{4s}{s^2+9} + \cancel{\frac{s(s^2+9)}{s^2+9}}\ s$$

$$Y = \frac{4s}{(s^2+1)(s^2+9)} + \frac{s}{s^2+1}$$

now do inverse LT to get $y \rightarrow \mathcal{L}^{-1}\{Y\} = y$
(table look up)

2nd term: $\dfrac{s}{s^2+1}$ is on table

1st term: $\dfrac{4s}{(s^2+1)(s^2+9)}$ is not table usually only has $\frac{\cdots}{s^2+1}$, $\frac{\cdots}{s^2+9}$ etc

we would like to split it up

$$\frac{4s}{(s^2+1)(s^2+9)} = \frac{\cdots}{s^2+1} + \frac{\cdots}{s^2+9}$$ partial fraction expansion

$$\frac{4s}{(s^2+1)(s^2+9)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+9}$$ numerator is one degree lower than denom

$$4s = (As+B)(s^2+9) + (Cs+D)(s^2+1)$$

$$\vdots$$

$$0s^3+0s^2+4s+0 = (A+C)s^3 + (B+D)s^2 + (A+9C)s + (B+9D)$$

$$\left.\begin{aligned} A+C&=0 \\ B+D&=0 \\ A+9C&=4 \\ B+9D&=0 \end{aligned}\right\} \cdots \quad A = -\tfrac{1}{2},\ B=0,\ C=+\tfrac{1}{2},\ D=0$$

Y becomes

$$Y = \underbrace{\frac{1}{2}\frac{s}{s^2+9} - \frac{1}{2}\frac{s}{s^2+1}}_{\text{decomposed}} + \frac{s}{s^2+1}$$

$$Y = \frac{1}{2}\frac{s}{s^2+9} + \frac{1}{2}\frac{s}{s^2+1}$$

$$y = \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{s}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{s}{s^2+1}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$\boxed{y = \frac{1}{2}\cos 3t + \frac{1}{2}\cos t}$$

LT w/ system

<u>example</u> $x' + 2y' + x = 0$ $\quad x(0) = 0 \quad y(0) = 1$

$x' - y' + y = 0$

LT both sides for both eqs

$$sX - x(0) + 2\left(sY - y(0)\right) + X = 0$$

$$sX - \left(sY - y(0)\right) + Y = 0$$

$$sX + 2sY + X = 2$$

$$sX - sY + Y = -1$$

$$(s+1)X + 2sY = 2$$

$$sX + (1-s)Y = -1$$

linear algebra problem

find X, Y

then $\mathcal{L}^{-1}\{X\}$

$\mathcal{L}^{-1}\{Y\}$

multiply eq. 1 by $\frac{1}{s+1}$

$$X + \frac{2s}{s+1} Y = \frac{2}{s+1}$$

multiply by $-s$, add to eq. 2

$$\left(\frac{-2s^2}{s+1} + 1 - s \right) Y = -\frac{2s}{s+1} - 1$$

$$\vdots$$

$$Y = \cdots \quad \rightarrow \quad y = \mathcal{L}^{-1}\{Y\}$$

x' from $x' - y' + y = 0$

then integrate to find x

<u>OR</u>

do the ~~sgst~~ linear algebra again for X

then $x = \mathcal{L}^{-1}\{X\}$

$\mathcal{L}\{f(t)\} = F(s)$

what is $\mathcal{L}\left\{ \int_0^t f(\tau)\,d\tau \right\} = ?$

use the same $\mathcal{L}\{y'\} = sY - y(0)$

$\vdots$

$$\boxed{\mathcal{L}\left\{ \int_0^t f(\tau)\,d\tau \right\} = \frac{F(s)}{s}}$$

$$\Updownarrow$$

$$\boxed{\mathcal{L}^{-1}\left\{ \frac{F(s)}{s} \right\} = \int_0^t f(\tau)\,d\tau}$$

choice: $\dfrac{1}{s(s^2+1)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+1}$

for example, $\mathcal{L}^{-1}\left\{ \dfrac{1}{s(s^2+1)} \right\}$

$$\mathcal{L}^{-1}\left\{ \frac{\overbrace{\frac{1}{s^2+1}}^{F}}{s} \right\} = \int_0^t \sin\tau \, d\tau = -\cos\tau \Big|_0^t$$

$$= -\cos t + 1$$