7.3 Translation of Laplace Transform

$\mathcal{L}\{f(t)\} = F(s)$

$F(s-a) = ?$ how does that affect $f(t)$? NOT $f(t-a)$

not shift in "time"

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\,e^{-st}\,dt = F(s)$$

$$F(s-a) = \int_0^\infty f(t)\,e^{-(s-a)t}\,dt = \int_0^\infty f(t)\,e^{-st}\,e^{at}\,dt$$

$$= \int_0^\infty [f(t)\,e^{at}]\,e^{-st}\,dt = \mathcal{L}\{f(t)e^{at}\}$$

so, a shift in the s domain is equivalent to multiplying by e^{at} in t domain sometimes simplifies LT

$$\mathcal{L}\{e^{3t}\underbrace{\sin 4t}_{f(t)}\} = F(s-3) \qquad \mathcal{L}\{\sin 4t\} = \frac{4}{s^2+16}$$

$$= \frac{4}{(s-3)^2+16}$$

on table: $\mathcal{L}\{1\} = \frac{1}{s}$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$$

e^{at} in t $\longleftrightarrow$ shift by a in s

inverse LT involving $e^{at}\cos bt$ and $e^{at}\sin bt$ can be annoying

$$\mathcal{L}^{-1}\left\{\frac{3s+5}{s^2-6s+25}\right\}$$

on table denominators are always sum/diff of squares

$$\mathcal{L}^{-1}\left\{\frac{3s+5}{s^2-6s+9+16}\right\}=\mathcal{L}^{-1}\left\{\frac{3s+5}{(s-3)^2+4^2}\right\}$$

$$\mathcal{L}\{e^{at}\cos bt\}=\frac{s-a}{(s-a)^2+b^2}$$

$$\mathcal{L}\{e^{at}\sin bt\}=\frac{b}{(s-a)^2+b^2}$$

$$=\mathcal{L}^{-1}\left\{3\cdot\frac{s}{(s-3)^2+4^2}+5\cdot\frac{1}{(s-3)^2+4^2}\right\}$$

want $s-3$ want 4

$$=3\,\mathcal{L}^{-1}\left\{\frac{s}{(s-3)^2+4^2}\right\}+5\,\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2+4^2}\right\}$$

$$=3\,\mathcal{L}^{-1}\left\{\frac{s\mp 3}{(s-3)^2+4^2}+\frac{3}{(s-3)^2+4^2}\right\}+\frac{5}{4}\mathcal{L}^{-1}\left\{\frac{4}{(s-3)^2+4^2}\right\}$$

$$=3\,\mathcal{L}^{-1}\left\{\frac{s-3}{(s-3)^2+4^2}\right\}+\frac{9}{4}\mathcal{L}^{-1}\left\{\frac{4}{(s-3)^2+4^2}\right\}+\frac{5}{4}\mathcal{L}^{-1}\left\{\frac{4}{(s-3)^2+4^2}\right\}$$

$$=\boxed{3e^{3t}\cos 4t+\frac{7}{2}e^{3t}\sin 4t}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2(s^2+4s+13)}\right\}$$

$$\frac{1}{(s+1)^2(s^2+4s+13)} = \underbrace{\frac{A}{s+1} + \frac{B}{(s+1)^2}}_{\text{repeated linear factors}} + \frac{Cs+D}{s^2+4s+13}$$

$$1 = A(s+1)(s^2+4s+13) + B(s^2+4s+13) + (Cs+D)(s+1)^2$$

$$\vdots$$

$$A = -\frac{1}{50}, \quad B = \frac{1}{10}, \quad C = \frac{1}{50}, \quad D = -\frac{1}{25}$$

$$= -\frac{1}{50}\underbrace{\frac{1}{s+1}}_{e^{-t}} + \frac{1}{10}\underbrace{\frac{1}{(s+1)^2}}_{te^{-t}} + \frac{1}{50}\underbrace{\frac{s}{s^2+4s+13}}_{\text{similar to}} - \frac{1}{25}\underbrace{\frac{1}{s^2+4s+13}}_{\text{previous example}}$$

$$= -\frac{1}{50}\frac{1}{s+1} + \frac{1}{10}\frac{1}{(s+1)^2} + \frac{1}{50}\frac{s}{(s+2)^2+3^2} - \frac{1}{25}\frac{1}{(s+2)^2+3^2}$$

$$= -\frac{1}{50}\frac{1}{s+1} + \frac{1}{10}\frac{1}{(s+1)^2} + \frac{1}{50}\frac{s+2}{(s+2)^2+3^2} - \frac{1}{50}\frac{2}{(s+2)^2+3^2} - \frac{1}{25}\frac{1}{(s+2)^2+3^2}$$

$$= -\frac{1}{50}\frac{1}{s+1} + \frac{1}{10}\frac{1}{(s+1)^2} + \frac{1}{50}\frac{s+2}{(s+2)^2+3^2} - \frac{2}{75}\frac{3}{(s+2)^2+3^2}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \qquad \mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$$

inverse LT

$$-\frac{1}{50}e^{-t} + \frac{1}{10}te^{-t} + \frac{1}{50}e^{-2t}\cos 3t - \frac{2}{75}e^{-2t}\sin 3t$$