7.4 Derivative, Integral, and Multiplication of LT's

last time: $F(s-a) = \mathcal{L}\{e^{at} f(t)\}$ $\qquad \frac{F(s)}{s} = \mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}$

$F'(s) = ?$ what does it do in t domain?

$$F(s) = \int_0^\infty f(t) e^{-st} dt$$

$$F'(s) = \frac{d}{ds}\int_0^\infty f(t) e^{-st} dt = \int_0^\infty f(t) \frac{d}{ds} e^{-st} dt$$

$$= \int_0^\infty f(t) e^{-st} \cdot -t\, dt = \int_0^\infty [-t f(t)] e^{-st} dt$$

$$= \mathcal{L}\{-t f(t)\}$$

$$\boxed{F'(s) = \mathcal{L}\{-t f(t)\}}$$

$$\boxed{\mathcal{L}^{-1}\{F'(s)\} = -t f(t)}$$

example $F(s) = \ln\left(\frac{s}{s^2-9}\right)$ $f(t) = ?$

NOT on the table

$$F(s) = \ln(s) - \ln(s^2-9)$$

$$F'(s) = \underbrace{\frac{1}{s}} - \underbrace{\frac{2s}{s^2-9}}$$

things like these are on the table

$$\mathcal{L}^{-1}\{F'\} = -t\, f(t)$$

$$\mathcal{L}^{-1}\{F'\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{2s}{s^2-9}\right\}$$

$$= 1 - 2\cosh(3t) = -t\, f(t)$$

$$\boxed{f(t) = \frac{1-2\cosh(3t)}{-t}}$$

now let's look at integral of LT $\int_s^\infty F(\sigma)\,d\sigma$ σ: integration variable

$$F(\sigma) = \int_0^\infty f(t)\,e^{-\sigma t}\,dt$$

$$\int_s^\infty F(\sigma)\,d\sigma = \int_s^\infty \int_0^\infty f(t)\,e^{-\sigma t}\,dt\,d\sigma$$

swap integration order

$$= \int_0^\infty \left(\int_s^\infty f(t)\,e^{-\sigma t}\,d\sigma \right) dt$$

$$= \int_0^\infty f(t) \cdot e^{-\sigma t} \cdot -\frac{1}{t} \Bigg|_{\sigma = s}^{\sigma = \infty} dt$$

$$= \int_0^\infty f(t) \left(\frac{e^{-st}}{t} \right) dt = \int_0^\infty \left[\frac{f(t)}{t} \right] e^{-st}\,dt$$

$$\boxed{\int_s^\infty F(\sigma)\,d\sigma = \mathcal{L}\left\{ \frac{f(t)}{t} \right\}}$$

<u>example</u> $\mathcal{L}\left\{\frac{\sin t}{t}\right\}$

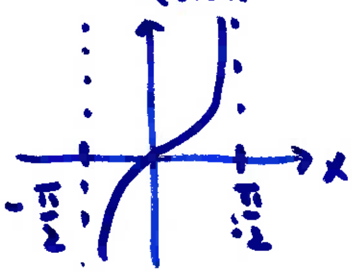
$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$$

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \mathcal{L}\{\sin t\}\,d\sigma$$

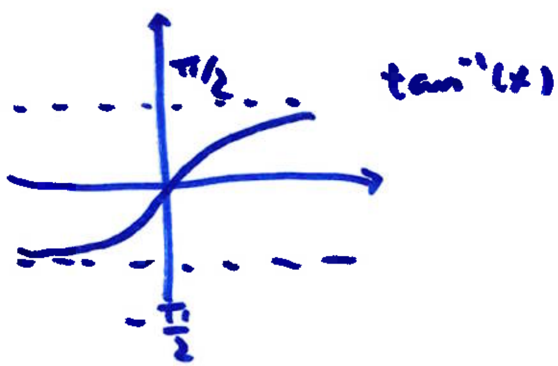
$$= \int_s^\infty \frac{1}{\sigma^2+1}\,d\sigma$$

$$= \tan^{-1}(\sigma)\Big|_{\sigma=s}^{\sigma=\infty} = \tan^{-1}(\infty) - \tan^{-1}(s)$$

tan x



$$\boxed{= \frac{\pi}{2} - \tan^{-1}(s) = \mathcal{L}\left\{\frac{\sin t}{t}\right\}}$$



$$\mathcal{L}\{f(t)+g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\} = F(s) + G(s)$$

but $\mathcal{L}\{f(t)\cdot g(t)\} \neq F(s)\,G(s)$

so what is it?

start with definition $\mathcal{L}\{h(t)\} = \int_0^\infty h(t)\,e^{-st}\,dt$

with $h(t) =$

(lengthy process)

$$\boxed{\begin{aligned}\mathcal{L}^{-1}\{F(s)\,G(s)\} &= \left.\begin{aligned}&\int_0^t f(\tau)\,g(t-\tau)\,d\tau \\ &= \int_0^t f(t-\tau)\,g(\tau)\,d\tau\end{aligned}\right\} \text{convolution integral} \\ &= f(t) * g(t)\end{aligned}}$$

multiplication in s domain is convolution in t domain

<u>example</u> $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$

we can do $\mathcal{L}^{-1}\left\{\frac{A}{s}+\frac{Bs+C}{s^2+1}\right\}$

or $\mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s}}_{F}\cdot\underbrace{\frac{1}{s^2+1}}_{G}\right\} = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t f(t-\tau)g(\tau)\,d\tau$

$f(t)=1$ $\quad$ $g(t)=\sin t$

$$\int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t 1\cdot\sin\tau\,d\tau$$

$$= -\cos\tau\Big|_{\tau=0}^{\tau=t} = -\cos t+1$$

$$= 1-\cos t$$

or $\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\,d\tau$