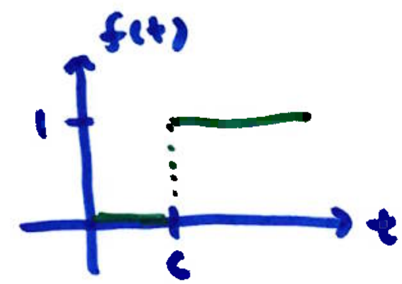
7.5 Piecewise Continuous Input Functions

$ay'' + by' + cy = f(t)$ $\quad$ $f(t)$ is discontinuous

"old" ways from 262/266 are not very good for discontinuous $f(t)$

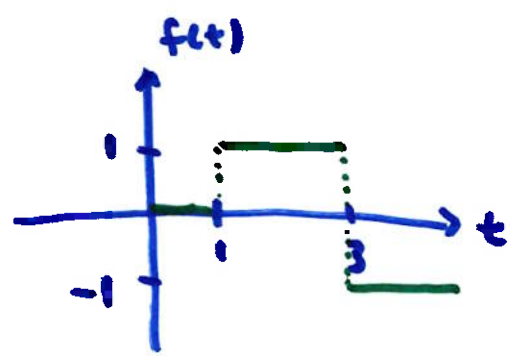
unit step function is used to model this kind of input

$$u(t-c) = \begin{cases} 1 & \text{if } t \geq c \\ 0 & \text{otherwise} \end{cases}$$

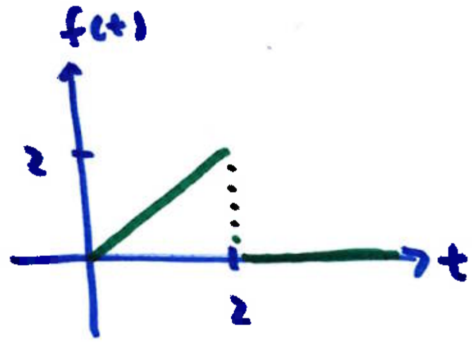


let's use it to model this

$$f(t) = \begin{cases} 0 & 0 \leq t < 1 \\ 1 & 1 \leq t < 3 \\ -1 & t \geq 3 \end{cases}$$



$$f(t) = 0 + \underbrace{u(t-1)}_{\substack{0 \text{ until} \\ t=1}} - 2\underbrace{u(t-3)}_{\substack{0 \text{ until} \\ t=3}}$$



$$f(t) = \begin{cases} t & 0 \le t < 2 \\ 0 & t \ge 2 \end{cases}$$

$$f(t) = t + \underbrace{u(t-2)}_{\substack{\text{activate} \\ \text{at } t \ge 2}}(-t)$$

turn this on $t \ge 2$

$$f(1) = 1 + (0)(-1) = 1$$

$$f(5) = 5 + (1)(-5) = 0$$

$$\mathcal{L}\{u(t-c)\} = \int_0^\infty u(t-c)\, e^{-st}\, dt = \int_0^c 0 \cdot e^{-st} dt + \int_c^\infty 1 \cdot e^{-st} dt$$

$$= -\frac{1}{s} e^{-st} \Big|_c^\infty = 0 - -\frac{1}{s} e^{-cs} = \frac{1}{s} e^{-cs}$$

$$\boxed{\mathcal{L}\{u(t-c)\} = e^{-cs}\cdot\frac{1}{s}}$$

$$\mathcal{L}\{u(t-c)\cdot f(t)\} = \int_0^c 0\cdot f(t)\cdot e^{-st}\,dt + \int_c^\infty f(t)e^{-st}\,dt$$

$$= \int_c^\infty f(t)e^{-st}\,dt$$

not 0 so not def. of LT

let $\tau = t-c$ $\quad d\tau = dt$

$$= \int_0^\infty f(\tau+c)e^{-s(\tau+c)}\,d\tau$$

$$= e^{-cs}\int_0^\infty f(\tau+c)e^{-s\tau}\,d\tau$$

$$= e^{-cs}\int_0^\infty f(t+c)e^{-st}\,dt$$

$$\boxed{\mathcal{L}\{u(t-c)\cdot f(t)\} = e^{-cs}\mathcal{L}\{\boxed{f(t+c)}\}}$$

$\mathcal{L}\{f(t+c)\}$ is LT of $f(t)$ Shifted to the LEFT by c t units

$f(t) = t - u(t-2)\cdot t$ → shift LEFT by 2 → change t to $t+2$

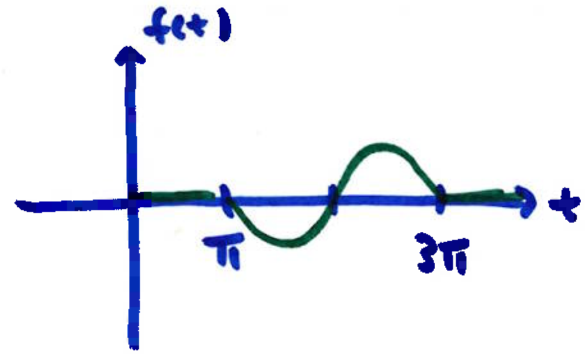
$$F(s) = \frac{1}{s^2} - e^{-2s}\cdot \mathcal{L}\{t+2\} = \boxed{\frac{1}{s^2} - e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right)}$$

transform $f(t)$ but t changed to $t+c$

table : $\mathcal{L}\{u(t-c)\,f(t-c)\} = e^{-cs}F(s)$ says the same

<u>example</u>

$$f(t) = \begin{cases} 0 & 0 \le t < \pi \\ \sin(t) & \pi \le t < 3\pi \\ 0 & t \ge 3\pi \end{cases}$$



$$f(t) = u(t-\pi)\cdot\sin(t) - u(t-3\pi)\cdot\sin(t)$$

$$F(s) = e^{-\pi s}\cdot\mathcal{L}\{\sin(t+\pi)\} - e^{-3\pi s}\mathcal{L}\{\sin(t+3\pi)\}$$

$$= e^{-\pi s}\mathcal{L}\{-\sin(t)\} - e^{-3\pi s}\mathcal{L}\{-\sin(t)\}$$

$$= \boxed{e^{-\pi s}\cdot\frac{-1}{s^2+1} + e^{-3\pi s}\cdot\frac{1}{s^2+1}}$$

inverse LT: reverse the steps above

inv. LT, then change $\underline{t}$ to ~~t~~ $\underline{t-c}$

example $F(s) = \underbrace{e^{-3s}}_{u(t-3)}\left(\frac{1}{s} + \frac{1}{s+4}\right)$

$\frac{1}{s}$: inverse directly is 1

$\frac{1}{s+4}$: inverse directly is e^{-4t} → next, change t to $t-c$ (shift RIGHT)

$$f(t) = u(t-3)\left(1 + e^{-4(t-3)}\right)$$

example $F(s) = \frac{1}{s^2} - 3\underbrace{e^{-2s}}_{u(t-2)}\frac{1}{s^2} + 2\underbrace{e^{-3s}}_{u(t-3)}\frac{1}{s^2}$

$\frac{1}{s^2} \to t$

$\frac{1}{s^2} \to t \to t-2$

$\frac{1}{s^2} \to t \to t-3$

$$f(t) = t - 3u(t-2)(t-2) + 2u(t-3)(t-3)$$

$$f(t) = \begin{cases} 0 & 0 \le t < \pi \\ \sin(t) & \pi \le t < 3\pi \\ 1 & t \ge 3\pi \end{cases}$$

$$= u(t-\pi)\sin(t) - u(t-3\pi)\sin(t) + u(t-3\pi)\cdot 1$$

Function	Transform	Function	Transform
$f(t)$	$F(s)$	e^{at}	$\dfrac{1}{s-a}$
$af(t)+bg(t)$	$aF(s)+bG(s)$	$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$f'(t)$	$sF(s)-f(0)$	$\cos kt$	$\dfrac{s}{s^2+k^2}$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$	$\sin kt$	$\dfrac{k}{s^2+k^2}$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	$\cosh kt$	$\dfrac{s}{s^2-k^2}$
$\int_0^t f(\tau)\,d\tau$	$\dfrac{F(s)}{s}$	$\sinh kt$	$\dfrac{k}{s^2-k^2}$
$e^{at}f(t)$	$F(s-a)$	$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$	$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	$\dfrac{1}{2k^3}(\sin kt-kt\cos kt)$	$\dfrac{1}{(s^2+k^2)^2}$
$tf(t)$	$-F'(s)$	$\dfrac{t}{2k}\sin kt$	$\dfrac{s}{(s^2+k^2)^2}$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$\dfrac{1}{2k}(\sin kt+kt\cos kt)$	$\dfrac{s^2}{(s^2+k^2)^2}$
$\dfrac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	$u(t-a)$	$\dfrac{e^{-as}}{s}$
$f(t)$, period p	$\dfrac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	$\delta(t-a)$	e^{-as}
1	$\dfrac{1}{s}$	$(-1)^{[\![t/a]\!]}$ (square wave)	$\dfrac{1}{s}\tanh\dfrac{as}{2}$
t	$\dfrac{1}{s^2}$	$\left[\!\left[\dfrac{t}{a}\right]\!\right]$ (staircase)	$\dfrac{e^{-as}}{s(1-e^{-as})}$
t^n	$\dfrac{n!}{s^{n+1}}$		
$\dfrac{1}{\sqrt{\pi t}}$	$\dfrac{1}{\sqrt{s}}$		
t^a	$\dfrac{\Gamma(a+1)}{s^{a+1}}$		