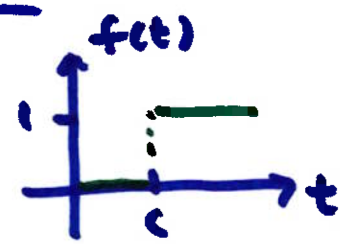
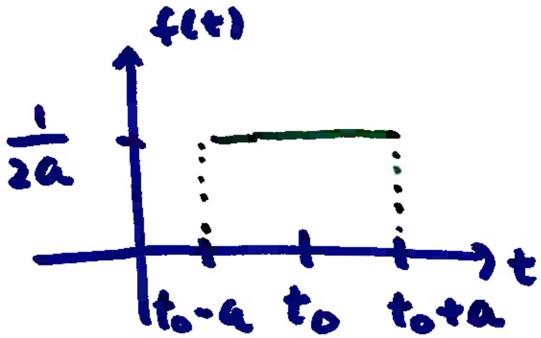
7.6 Impulse Function

$$u(t-c) = \begin{cases} 1 & t \geq c \\ 0 & \text{else} \end{cases}$$



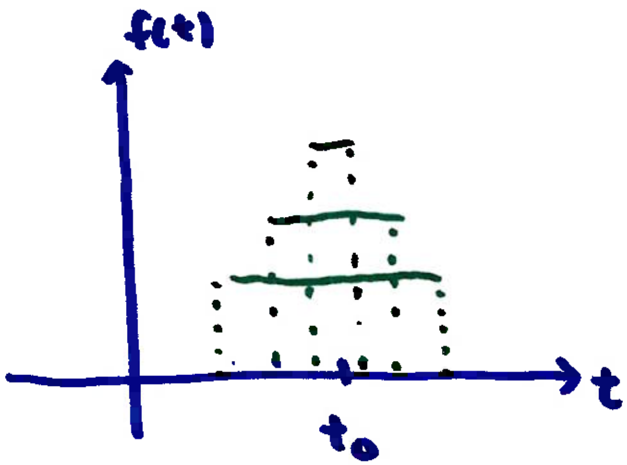
turn on, then turn off



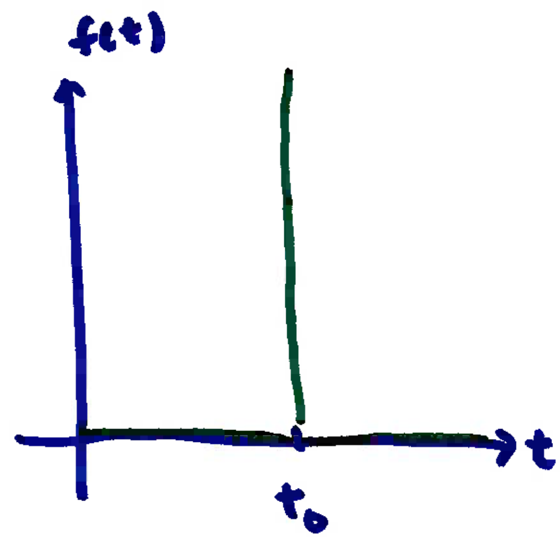
duration: $2a$

area is 1 so height is $\frac{1}{2a}$

now take limit as $a \to 0$



$\rightarrow$



impulse function: $\delta(t-t_0)$ 0 except at $t=t_0$

area under is 1

use to model short duration input (hitting a golf ball, for example)

$$\mathcal{L}\{\delta(t-t_0)\} = ?$$

$$\mathcal{L}\left\{\lim_{a\to 0}\left[\frac{u(t-(t_0-a))}{2a} - \frac{u(t-(t_0+a))}{2a}\right]\right\}$$

$$= \frac{1}{2a}\lim_{a\to 0}\left(\int_{t_0-a}^{\infty} e^{-st}\,dt - \int_{t_0+a}^{\infty} e^{-st}\,dt\right)$$

$$= \lim_{a\to 0}\frac{1}{2a}\left(-\frac{1}{s}e^{-st}\Big|_{t_0-a}^{\infty} + \frac{1}{s}e^{-st}\Big|_{t_0+a}^{\infty}\right)$$

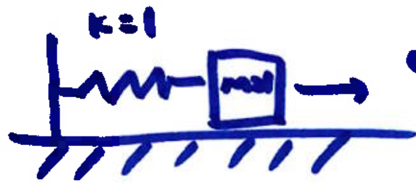
$$= \lim_{a\to 0}\frac{1}{2a}\left(\frac{1}{s}e^{-s(t_0-a)} - \frac{1}{s}e^{-s(t_0+a)}\right)$$

$$= \cdots = e^{-t_0 s}$$

$$\boxed{\mathcal{L}\{\delta(t-t_0)\} = e^{-t_0 s}}$$

what function has a LT of 1 ? $\delta(t) \rightarrow$ impulse at $t=0$

<u>example</u> $y'' + y = 1 + \delta(t-\pi)$ $y(0) = y'(0) = 0$



constant force of 1
then an impulse at $t=\pi$

LT both sides

$$s^2 Y - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} + Y = \frac{1}{s} + e^{-\pi s}$$

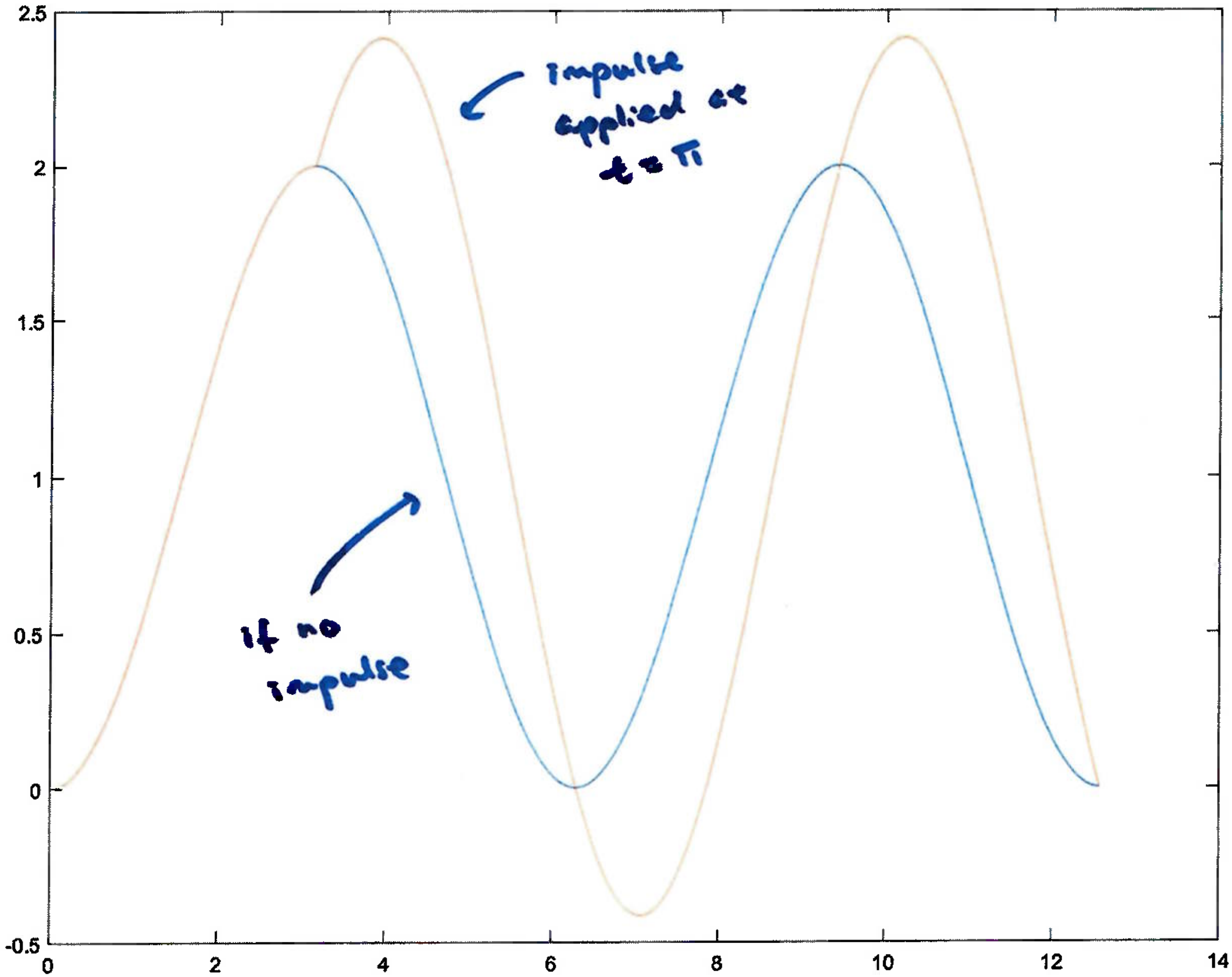
$$(s^2+1)Y = \frac{1}{s} + e^{-\pi s}$$

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

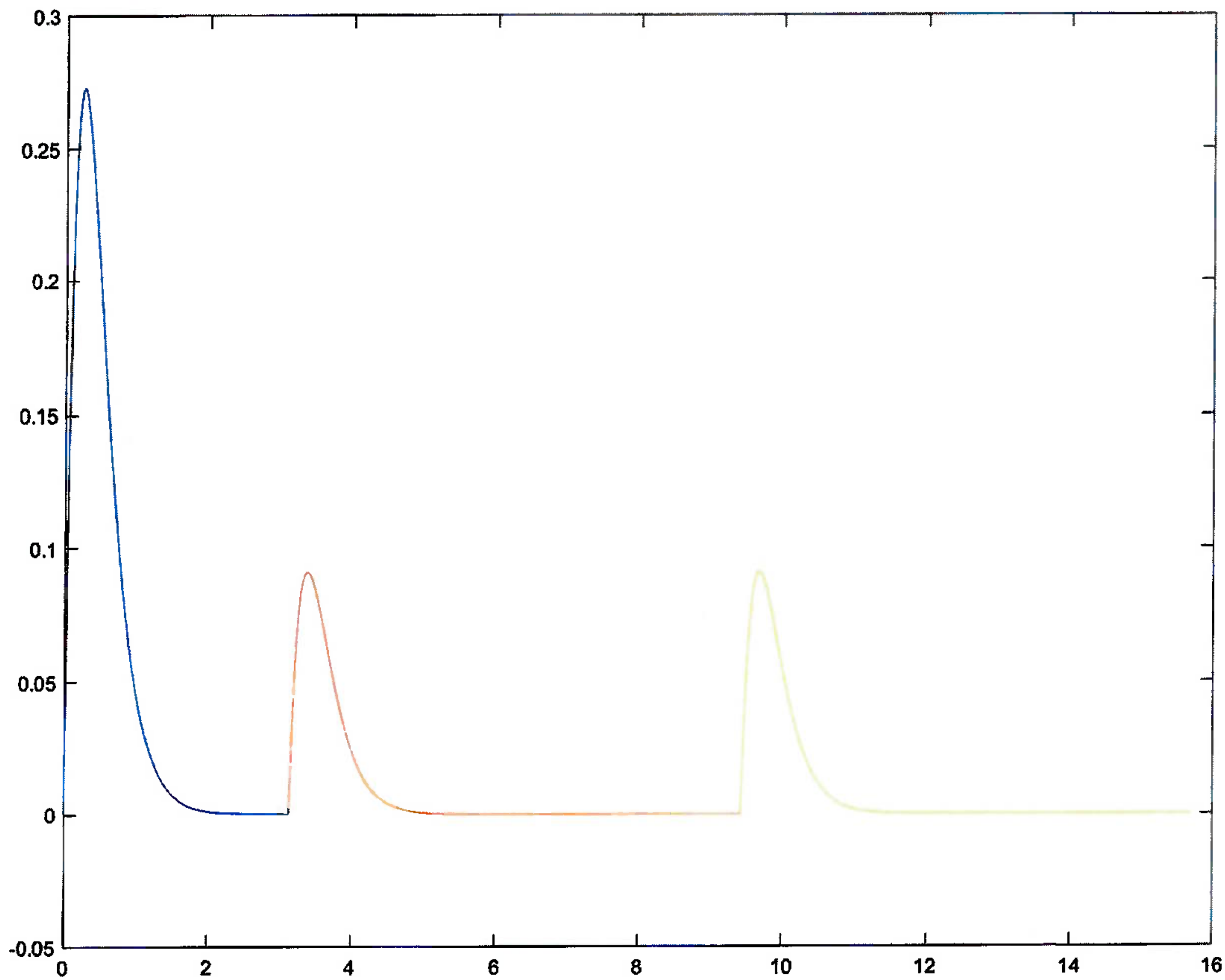
$$Y = \underbrace{\frac{1}{s(s^2+1)}}_{1-\cos(t)} + \underbrace{\overset{u(t-\pi)}{e^{-\pi s}} \, \underbrace{\frac{1}{s^2+1}}_{\sin(t)}}$$

but shifted due to $e^{-\pi s}$ in front
$t \rightarrow t+\pi$ or $\underline{\underline{t-\pi}}$?

$$y = 1 - \cos(t) + u(t-\pi)\,(\underbrace{\sin(t-\pi)}_{-\sin(t)})$$



2.5
2
1.5
1
0.5
0
-0.5
0
2
4
6
8
10
12
14
impulse applied at t = π
if no impulse



0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
0
2
4
6
8
10
12
14
16

$$y = 1-\cos(t) + u(t-\pi)(-\sin(t))$$

$$= \begin{cases} 1-\cos(t) & 0 \le t < \pi \\ 1-\cos(t)-\sin(t) & t \ge \pi \end{cases}$$

<u>example</u> $y'' + 8y' + 17y = \delta(t-\pi) + \delta(t-3\pi) \qquad y(0)=0 \quad y'(0)=3$

$$s^2Y - sy(0) - y'(0) + 8sY - 8y(0) + 17Y = e^{-\pi s} + e^{-3\pi s}$$

$$(s^2+8s+17)Y = 3 + e^{-\pi s} + e^{-3\pi s}$$

$$e^{-4t}\sin(t)$$

$$Y = \frac{3}{s^2+8s+17} + e^{-\pi s}\frac{1}{s^2+8s+17} + e^{-3\pi s}\frac{1}{s^2+8s+17}$$

$$(s+4)^2+1$$

$$y = 3e^{-4t}\sin(t) + u(t-\pi)e^{-4(t-\pi)}\sin(t-\pi)$$
$$+ u(t-3\pi)e^{-4(t-3\pi)}\sin(t-3\pi)$$

$$= 3e^{-4t}\sin(t) + \underset{u(t-\pi)}{\wedge} e^{4\pi}e^{-4t}\sin(t) + u(t-3\pi)e^{4\pi}e^{-4t}\sin(t)$$

$$y'' + ay' + by = f(t) \qquad y(0) = y'(0) = 0$$

$$(s^2 + as + b)Y = F$$

$$Y = \frac{1}{s^2+as+b} \cdot F \qquad \mathcal{L}^{-1}\left\{\frac{1}{s^2+as+b}\right\} = h(t)$$

$$\frac{Y}{F} = \frac{1}{s^2+as+b}$$

Transfer function

→ system's impulse response

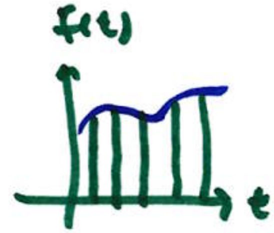
convolution

$$y(t) = \int_0^t h(t-\tau) f(\tau)\,d\tau = \int_0^t h(\tau) f(t-\tau)\,d\tau$$

treats $f(t)$ as a bunch of impulses

Sums up all impulse responses at

t varies



(convolution → Duhamel's principle)