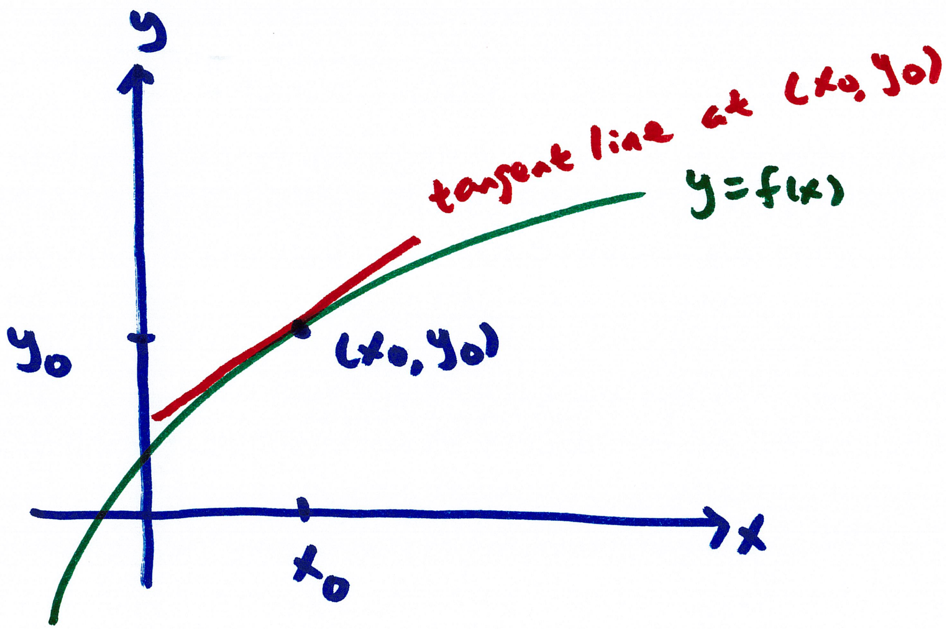
# 2.4 Euler's Method

numerical method : not getting a function $y = f(x)$
but as many points as we want
$(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots$

Euler's method is very simple but is the foundation of more sophisticated methods

→ also known as tangent line method

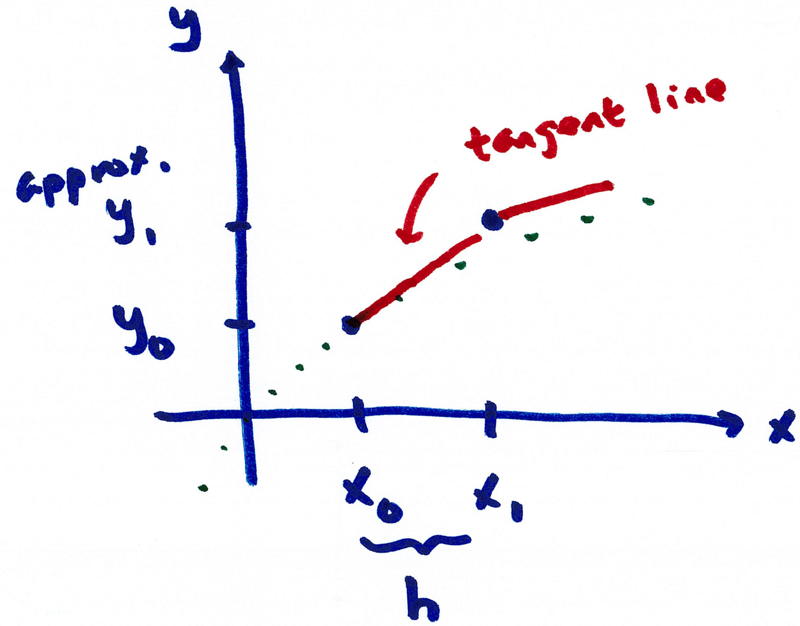



near $(x_0, y_0)$
tangent line $\approx$ true curve

in calculus, we know $y = f(x)$, we construct tangent line by finding $f'(x)$

when solving $y' = f(x, y)$, we don't have $y$

but we have $y'$ at any point




initial condition $(x_0, y_0)$

travel on that tangent line

to $x_1 = x_0 + h$

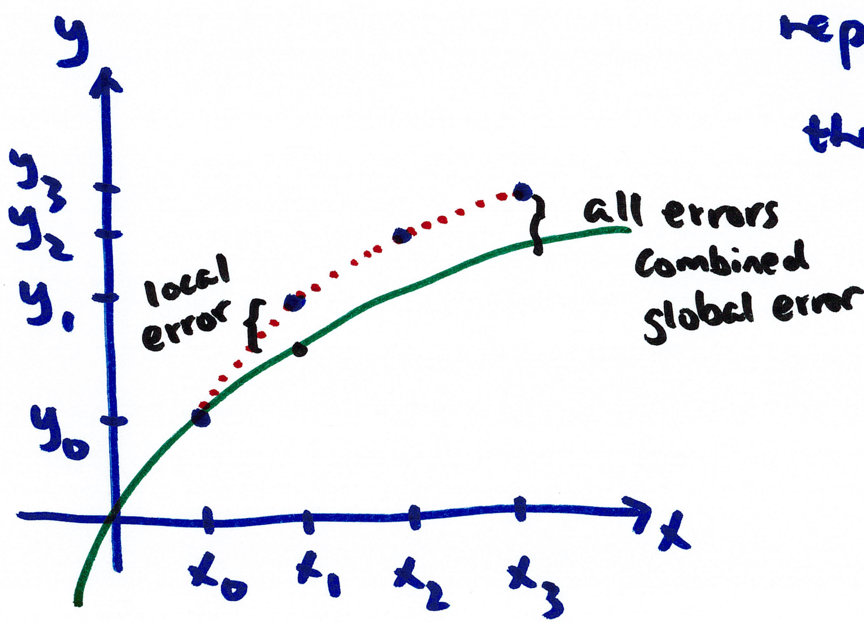
step size

then make a new tangent

line at ~~y~~ $(x_1, y_1)$

repeat until we reached

the destination $x$



## Algorithm   $y' = f(x, y)$

pick a fixed step size $h$

initial point $\left.\begin{matrix} x_0 \\ y_0 \end{matrix}\right\}$ known.

make a tangent line at $(x_0, y_0)$   $y - y_0 = f(x_0, y_0)(x - x_0)$

new $x$: $x_1 = x_0 + h$

estimate $y$: $y_1 = y_0 + f(x_0, y_0) h$

repeat: $x_2 = x_1 + h$

$y_2 = y_1 + f(x_1, y_1) h$

repeat until reach destination $X$

$$\boxed{y_n = \underset{\text{prev. } y}{y_{n-1}} + \underbrace{f(x_{n-1}, y_{n-1})}_{\text{slope at the previous point}} h}$$

<u>example</u> $y' = 2y - 3x \quad y(0) = 1$

estimate $y(0.5)$ with a step size of $h = 0.25$

$x_0 = 0$

$y_0 = 1$

$x_1 = x_0 + h = 0 + 0.25 = 0.25$

$y_1 = y_0 + f(x_0, y_0)h$

$= 1 + [2(y_0) - 3(x_0)](0.25)$

$= 1 + [2(1) - 3(0)](0.25) = 1.5$

$x_2 = x_1 + h = 0.25 + 0.25 = 0.5$ destination → final step

$y_2 = y_1 + f(x_1, y_1)h$

$= 1.5 + [2(1.5) - 3(0.25)](0.25) = \boxed{2.0625}$

Our estimate of $y(0.5)$

$y' = 2y - 3x \quad y(0) = 1$ is one that we can solve

let's solve it and see the accuracy

1st-order linear: solvable by using integrating factor

$$y' - 2y = -3x$$

integrating factor: $\mu = e^{\int -2\,dx} = e^{-2x}$

$$e^{-2x}(y' - 2y) = -3xe^{-2x}$$

$$\frac{d}{dx}(e^{-2x}y) = -3xe^{-2x}$$

$$e^{-2x}y = \int -3xe^{-2x}\,dx$$

$$\vdots$$

$$y = \frac{3}{4}(2x+1) + \frac{1}{4}e^{2x}$$

true $y(0.5) = 2.1796$

our estimate: 2.0625 (5% error)

to improve, shrink step size (more steps) if we used $h = 0.01$ (25 steps)

estimated $y(0.5) = 2.1729$ (0.3 % error)

in practice, we don't know the true $y$ (otherwise why use Euler's method?)

so how to decide if $h$ we picked is good?

→ keep decreasing $h$ until the estimate doesn't change much

→ means we are very close to the true $y$

using $y' = 2y - 3x$ $y(0) = 1$ as an example to estimate $y(1)$

$h = 0.5$ $y(1) = 3.25$

$h = 0.05$ $y(1) = 3.932$

big change, suggesting $h = 0.5$ is bad

$h = 0.01$ $y(1) = 4.061$

$h = 0.001$ $y(1) = 4.094$

still movement, suggesting improvement possible

$h = 0.0005$ $y(1) = 4.095$

small improvement, suggesting accuracy is good enough