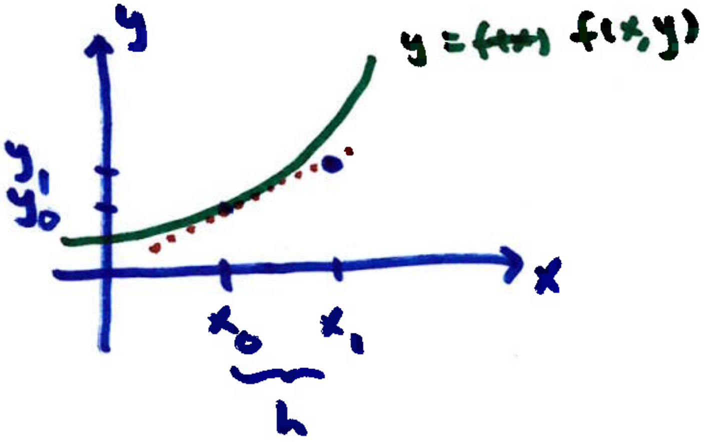
2.5 Improved Euler's Method

Euler: use tangent line at previous point

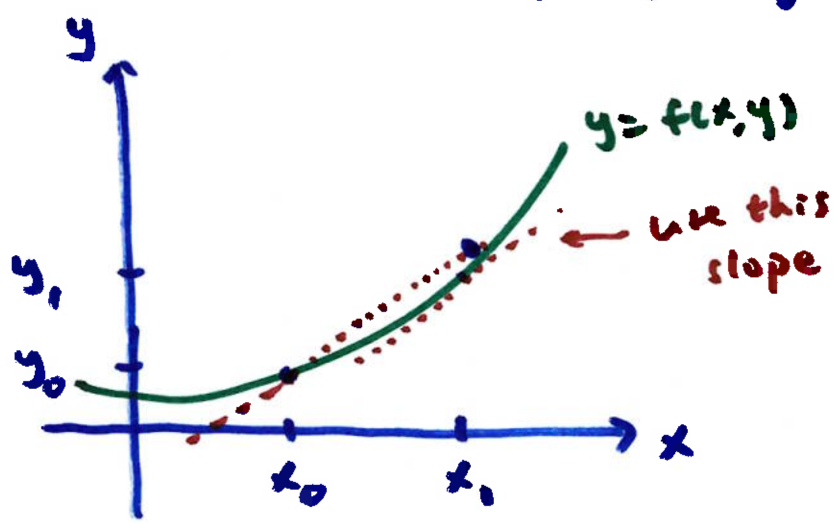


$$y_{n+1} = y_n + f(x_0, y_0)h$$

if $f(x)$ changes rapidly, then small h is needed

Small h: many steps
numerical instability

Backward Euler: use slope of tangent line at later point



$$y_{n+1} = y_n + \overbrace{f(x_{n+1}, y_{n+1})}^{\text{slope at new point}}h$$

we need y_{n+1} to find y_{n+1} ?!

Backward: implicit method (need to find y_{n+1} somehow)

$$y_{n+1} = y_n + f(x_{n+1}, y_{n+1})h$$

to find y_{n+1}, we need to solve an algebraic equation (with a root finding technique)

example $y' = 2y - 3x \quad y(0) = 1$

use $h = 0.1$ to estimate $y(0.1)$

$x_0 = 0$

$y_0 = 1$

$x_1 = x_0 + h = 0.1$

$$y_1 = y_0 + f(\underbrace{x_1, y_1}_{\text{new location}})h$$

$$= 1 + [2(y_1) - 3(0.1)](0.1)$$

$$y_1 = 1 + 0.2y_1 - 0.03$$

$$0.8y_1 = 0.97$$

$$\boxed{y_1 = 1.2125}$$

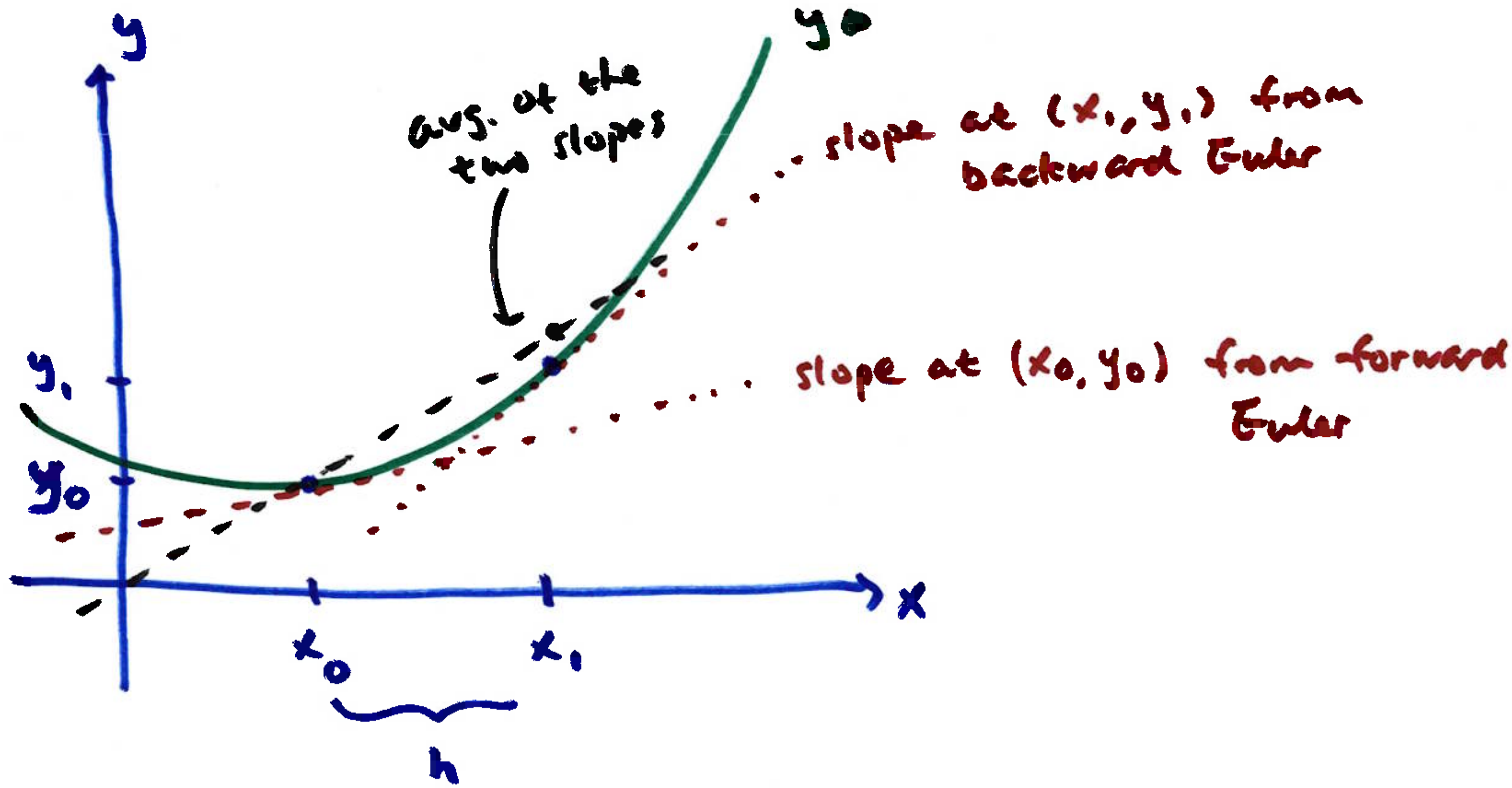
this algebraic eq. gives y_1

linear (like this one) is solvable by hand

nonlinear: with (e.g.) Newton's Method

now we combine the two to form Improved Euler's Method (also called Heun's method)

→ average the two slopes



$$\boxed{y_{n+1} = y_n + \frac{1}{2}\Big[\underbrace{f(x_n, y_n)}_{\text{forward}} + \underbrace{f(x_{n+1}, y_{n+1})}_{\text{backward}}\Big] h}$$

practical issue: y_{n+1} on the right

Solution: use a regular/forward Euler for the y_{n+1} on the right

$$y_{n+1} = y_n + \frac{1}{2}\left[f(x_n, y_n) + f(x_{n+1}, \underbrace{y_n + f(x_n, y_n)h}_{\text{forward Euler}}\right] h$$

in practice, we do this in two stages

<u>example</u> $y' = 2y - 3x \quad y(0) = 1$

$h = 0.05$ estimate $y(0.1)$

$x_0 = 0$

$y_0 = 1$

$x_1 = x_0 + h = 0.05$

first, use forward Euler to find y_1

$y_1 = y_0 + f(x_0, y_0)h$

$= 1 + [2(1) - 3(0)](0.05) = 1.1$ predictor step

now use that y_1 in Improved Euler

$y_1 = y_0 + \frac{1}{2}[f(x_0, y_0) + f(x_1, y_1)]h$ corrector step

$= 1 + \frac{1}{2}[(2 \cdot 1 - 3 \cdot 0) + (2 \cdot 1.1 - 3 \cdot 0.05)](0.05)$

$= 1.10125 = y(0.05)$

$x_2 = x_1 + h = 0.1$ (target)

predictor $y_2 = y_1 + f(x_1, y_1) h$

$= 1.10125 + (2 \cdot 1.10125 - 3 \cdot 0.05)(0.05) = \underline{1.203875}$

now refine it using Improved Euler

$y_2 = y_1 + \frac{1}{2}\left[f(x_1, y_1) + f(x_2, y_2)\right] h$

$= 1.10125 + \frac{1}{2}\Big[(2 \cdot 1.10125 - 3 \cdot 0.05)$

$+ (2 \cdot 1.203875 - 3 \cdot 0.1)\Big](0.05)$

$$\boxed{y_2 = 1.20525625}$$

from forward Euler: $y = 1.2025$

true value : $y = 1.20535$

better accuracy but more steps (each step in improved is two sub steps)

Improved Euler is actually one method in the Runge-Kutta (RK) family → RK2 (Runge-Kutta order 2 = Improved Euler)