9.1 Periodic Functions and Fourier Series

Taylor series : $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$f(x)$ as combinations of x^n

for example, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \qquad |x| < 1$$

Taylor / Maclaurin: x^n as the basis functions

$\frac{1}{2!}$ part of x^2

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

one part of x

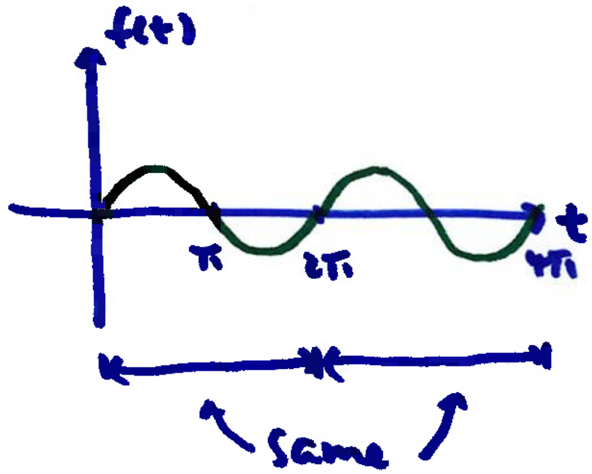
$\frac{1}{3!}$ part of x^3

Fourier series : $f(t)$ that is periodic

$f(t)$ can be expressed as an infinite sum of $\cos(nt)$ and $\sin(nt)$

periodic : repeats over some interval of t

$f(t) = f(t+T)$ T: period



here, $T = 2\pi$

if T is a period, so is $2T$, $3T$, $4T$, ...

the shortest time to repeat is called the fundamental period for example, $\sin(t)$ has fundamental period of 2π

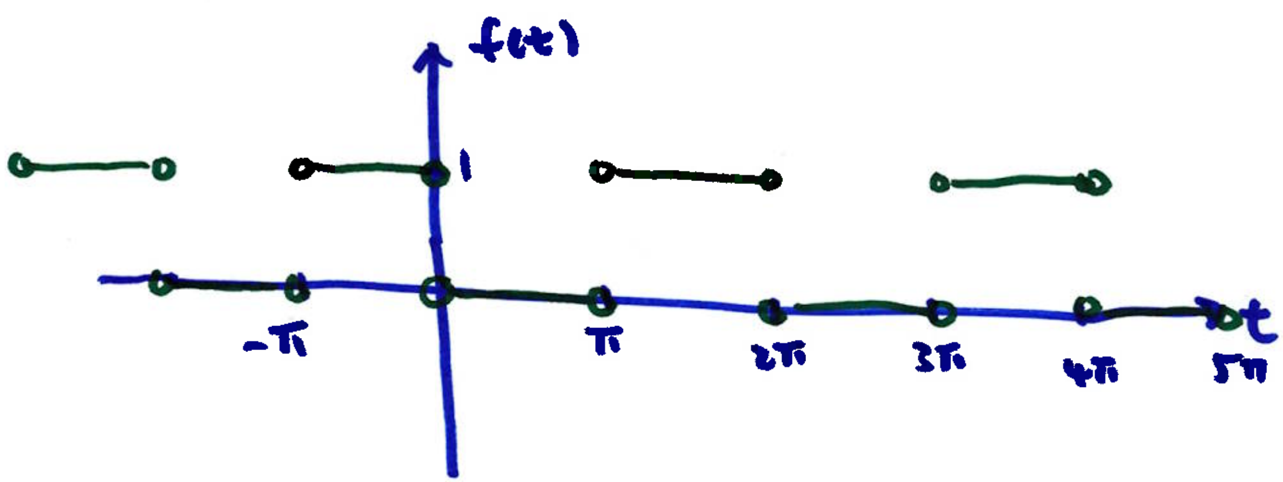
but its period is also $4\pi, 6\pi, 8\pi, \ldots$

$\sin(3t)$ has fundamental period of $\frac{2\pi}{3}$

but 2π is also a period

this is also a periodic function:

$$f(t) = \begin{cases} 1 & \text{if } -\pi < t < 0 \\ 0 & \text{if } 0 < t < \pi \end{cases}$$ with period of 2π



periodic and piecewise smooth

↓

express as Fourier series

→ combo of $\cos(nt)$ and $\sin(nt)$ $n = 0, 1, 2, 3, \ldots$

Fourier series

$\cos(0t) = 1$

$$f(t) = \boxed{\tfrac{1}{2}} a_0 + a_1\cos(t) + a_2\cos(2t) + a_3\cos(3t) + \cdots$$
$$+ b_1\sin(t) + b_2\sin(2t) + b_3\sin(3t) + \cdots$$

goes with a_0 only

$$\boxed{f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos(nt) + b_n\sin(nt)}$$

$\boxed{\text{f(t) is period with period } 2\pi}$

$-\pi < t < \pi$

$a_n = ?$ $b_n = ?$

to find the constants, we do this:

$$f(t) = \frac{1}{2}a_0 + a_1\cos(t) + a_2\cos(2t) + \cdots + b_1\sin(t) + b_2\sin(2t) + \cdots$$

to find a_n, multiply both sides by ~~a_n~~ $\cos(nt)$ and integrate over period

$$f(t)\cos(nt) = \frac{1}{2}a_0\cos(nt) + a_1\cos(t)\cos(nt) + a_2\cos(2t)\cos(nt) + \cdots$$
$$+ b_1\sin(t)\cos(nt) + b_2\sin(2t)\cos(nt) + \cdots$$

integrate from $t = -\pi$ to $t = \pi$

$$\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \int_{-\pi}^{\pi} a_0 \cos(nt)\,dt + \int_{-\pi}^{\pi} a_1 \cos(t)\cos(nt)\,dt + \cdots$$

$$+ \int_{-\pi}^{\pi} b_1 \sin(t)\cos(nt)\,dt + \int_{-\pi}^{\pi} b_2 \sin(2t)\cos(nt)\,dt + \cdots$$

we can show
$$\int_{-\pi}^{\pi} \cos(nt)\cos(mt)\,dt = \begin{cases} \pi & \text{if } n = m \ (n = m \neq 0) \\ 0 & \text{if } n \neq m \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(2t)\cos(3t)\,dt = 0$$

$$\int_{-\pi}^{\pi} \cos(2t)\cos(2t)\,dt = \pi$$

if $n = m = 0$

$$\underbrace{\int_{-\pi}^{\pi} \cos(nt)\cos(mt)\,dt}_{a_0 \text{ situation}} = \underline{\underline{2\pi}}$$

this is why a_0 gets a $\frac{1}{2}$

also, $\int_{-\pi}^{\pi} \cos(nt)\sin(mt)\,dt = 0$ for all n, m

so, the integrals reduce to $\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = a_n \cdot \pi$

this gives us $\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt}$ $n = 0, 1, 2, 3, \ldots$

to find b_n, repeat the process but multiply by $\sin(nt)$

we know $\int_{-\pi}^{\pi} \sin(nt)\sin(mt)\,dt = \begin{cases} \pi & \text{if } n = m \text{ (except } n = m = 0) \\ 0 & \text{if } n \neq m \end{cases}$

this gives us $\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt}$ $n = 1, 2, 3, \ldots$

try on $f(t) = \begin{cases} 1 & \text{if } -\pi < t < 0 \\ 0 & \text{if } 0 < t < \pi \end{cases}$ period 2π

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 1\cdot dt + \frac{1}{\pi}\int_{0}^{\pi} 0\cdot dt$$

$$= 1$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 1\cdot\cos(nt)\,dt + \frac{1}{\pi}\int_{0}^{\pi} 0\cdot\cos(nt)\,dt$$

$$= \frac{1}{\pi}\,\frac{1}{n}\sin(nt)\Big|_{-\pi}^{0} = \frac{1}{\pi}\cdot\left(\frac{1}{n}\sin(0) - \frac{1}{n}\sin(-n\pi)\right) = 0$$

all $a_n = 0$
except $a_0 = 1$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} \sin(nt)\,dt$$

$$= \frac{1}{\pi}\cdot -\frac{1}{n}\cos(nt)\Big|_{-\pi}^{0} = \frac{1}{\pi}\left(-\frac{1}{n}+\frac{1}{n}\underbrace{\cos(-n\pi)}_{=\begin{cases}1 & \text{if } n \text{ even}\\ -1 & \text{if } n \text{ odd}\end{cases}}\right) \quad n=1,2,3,\ldots$$

$$= \frac{1}{\pi}\left(-\frac{1}{n}+\frac{1}{n}(-1)^n\right)$$

$$= \begin{cases} \frac{1}{\pi}\left(-\frac{1}{n}+\frac{1}{n}\right) & n \text{ even} \\ \frac{1}{\pi}\left(-\frac{1}{n}-\frac{1}{n}\right) & n \text{ odd} \end{cases}$$

$$= \begin{cases} 0 & n \text{ even} \\ -\frac{2}{n\pi} & n \text{ odd} \end{cases}$$