9.1 Fourier Series (part 2)

$f(t)$ periodic with period 2π defined on $-\pi < t < \pi$

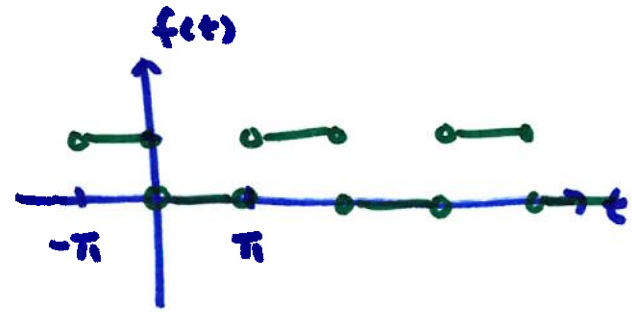
can be written as $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(nt) + b_n \sin(nt)\right]$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt \qquad n = 0, 1, 2, 3, \ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

example from last time

$$f(t) = \begin{cases} 1 & \text{if } -\pi \le t < 0 \\ 0 & \text{if } 0 \le t \le \pi \end{cases}$$



from last time: $a_0 = 1$, $a_n = 0$ for $n \ge 1$

$$b_n = \frac{-1 + (-1)^n}{n\pi} = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

Fourier series representation:

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt)$$

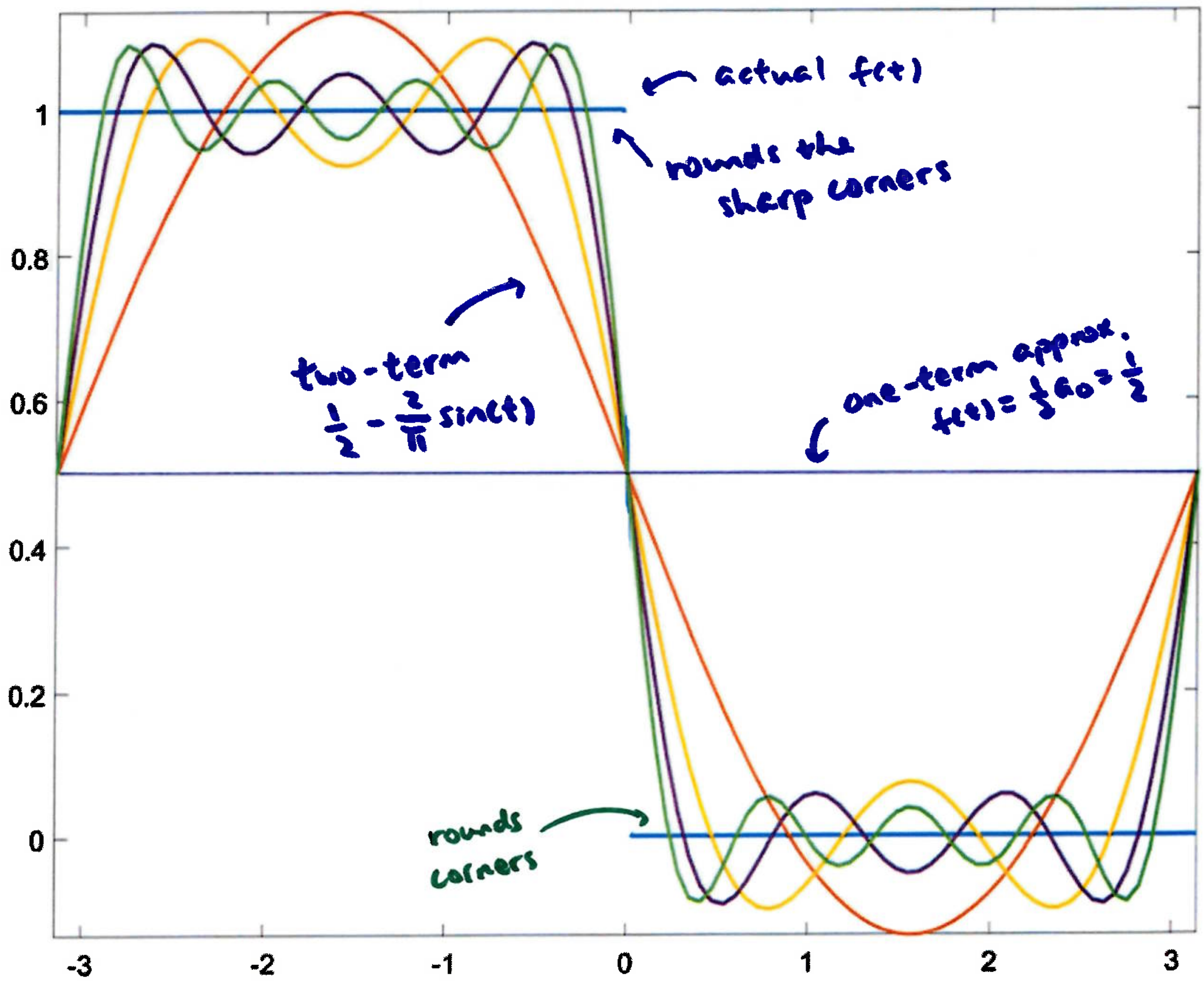
$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-1+(-1)^n}{n\pi} \sin(nt)$$ write out a few

$$= \frac{1}{2} - \frac{2}{\pi}\sin(t) - \frac{2}{3\pi}\sin(3t) - \frac{2}{5\pi}\sin(5t) - \cdots$$

like with Taylor series, the more terms we include, the closer the Fourier series representation will look like the real $f(t)$

$n \to \infty$, the approx. gets better

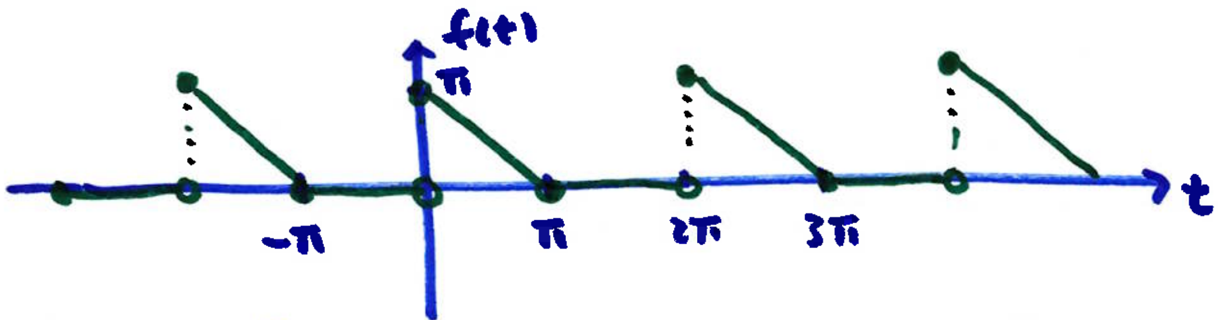
eventually converge to the actual $f(t)$



1
0.8
0.6
0.4
0.2
0
-3
-2
-1
0
1
2
3
actual f(t)
rounds the sharp corners
two-term
$\frac{1}{2} - \frac{2}{\pi}\sin(t)$
one-term approx.
$f(t) = \frac{1}{2}a_0 = \frac{1}{2}$
rounds corners

example

$$f(t) = \begin{cases} 0 & -\pi \le t < 0 \\ \pi - t & 0 \le t < \pi \end{cases} \qquad \text{period } 2\pi$$



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{0}^{\pi}(\pi - t)\,dt = \cdots = \frac{1}{2}\pi$$

$$a_n = \frac{1}{\pi}\int_0^{\pi}(\pi - t)\cos(nt)\,dt \quad \text{by parts} \quad \begin{matrix} u = \pi - t & dv = \cos(nt)\,dt \\ du = -dt & v = \frac{1}{n}\sin(nt) \end{matrix}$$

$$= \frac{1}{\pi}\left(\overbrace{(\pi - t)\frac{1}{n}\sin(nt)\Big|_0^{\pi}}^{0 \quad \sin(n\pi) = 0} + \int_0^{\pi}\frac{1}{n}\sin(nt)\,dt \right)$$

$$= -\frac{1}{n^2\pi}\cos(nt)\Big|_0^{\pi} = \frac{-1}{n^2\pi}\Big(\underbrace{\cos(n\pi)}_{(-1)^n} - 1\Big) = \frac{1}{n^2\pi}\left(1 - (-1)^n\right)$$

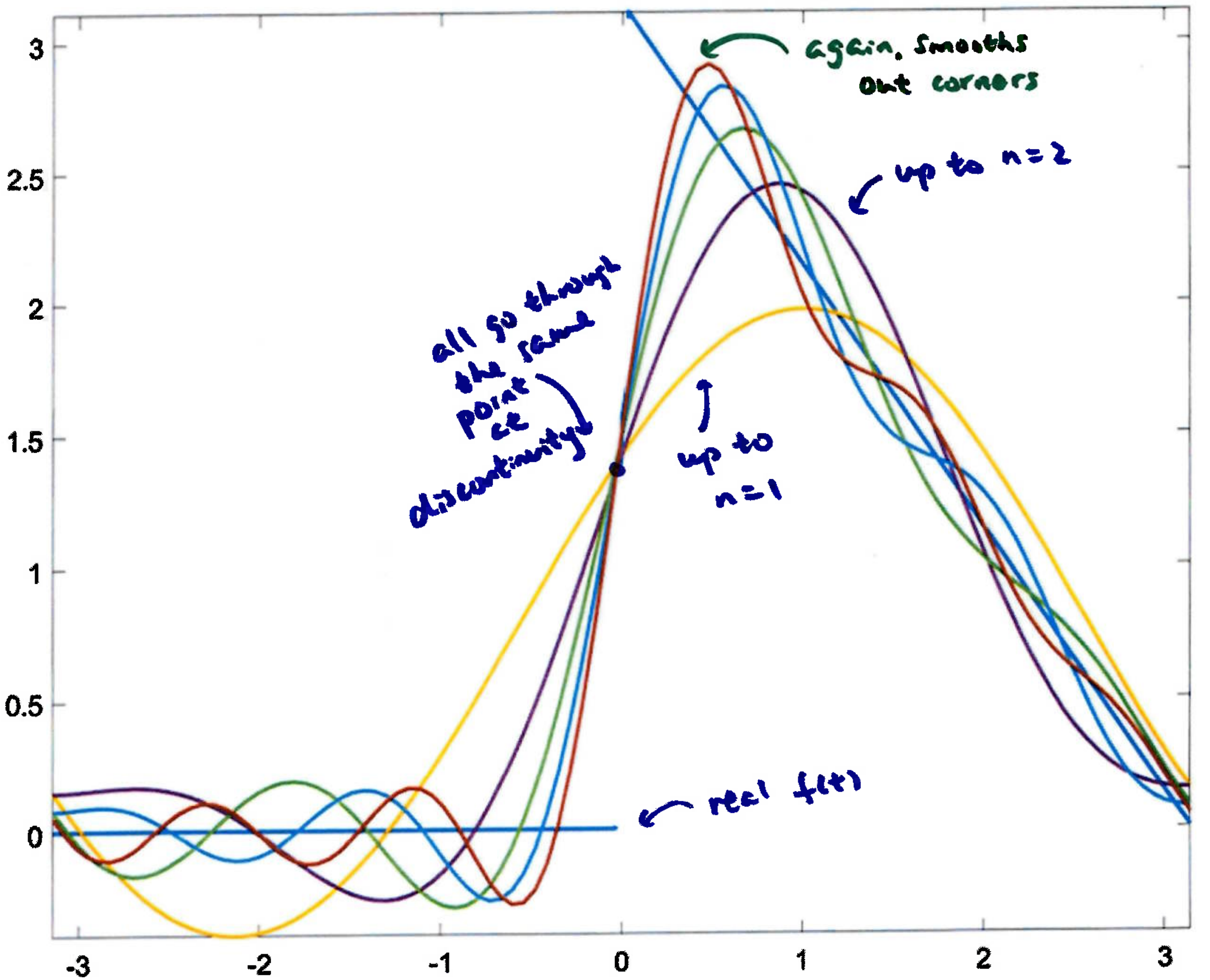
$$b_n = \frac{1}{\pi}\int_0^{\pi} (\pi - t)\sin(nt)\,dt = \cdots = \frac{1}{n}$$

Series representation:

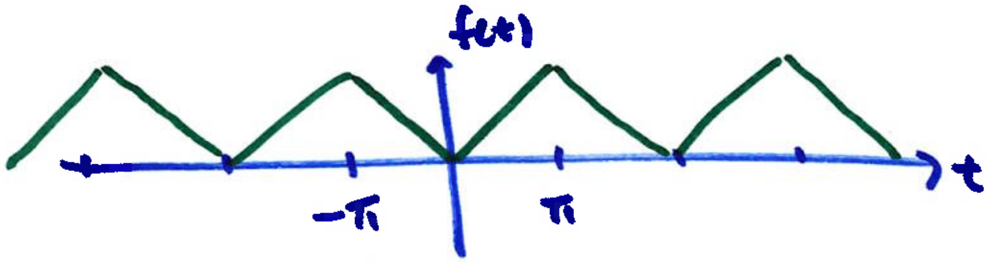
$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt)$$

$$= \frac{1}{4}\pi + \sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi}\left(1-(-1)^n\right)\cos(nt) + \frac{1}{n}\sin(nt)\right]$$

$$= \frac{1}{4}\pi + \underbrace{\frac{2}{\pi}\cos(t) + \sin(t)}_{n=1} + \underbrace{\frac{1}{2}\sin(2t)}_{n=2} + \underbrace{\frac{2}{9\pi}\cos(3t) + \frac{1}{3}\sin(3t)}_{n=3} + \cdots$$


again, smooths out corners
up to n=2
all go through the same point at discontinuity
up to n=1
real f(t)
3
2.5
2
1.5
1
0.5
0
-3
-2
-1
0
1
2
3

<u>example</u> $f(t) = |t| \quad -\pi \le t \le \pi \quad$ period 2π

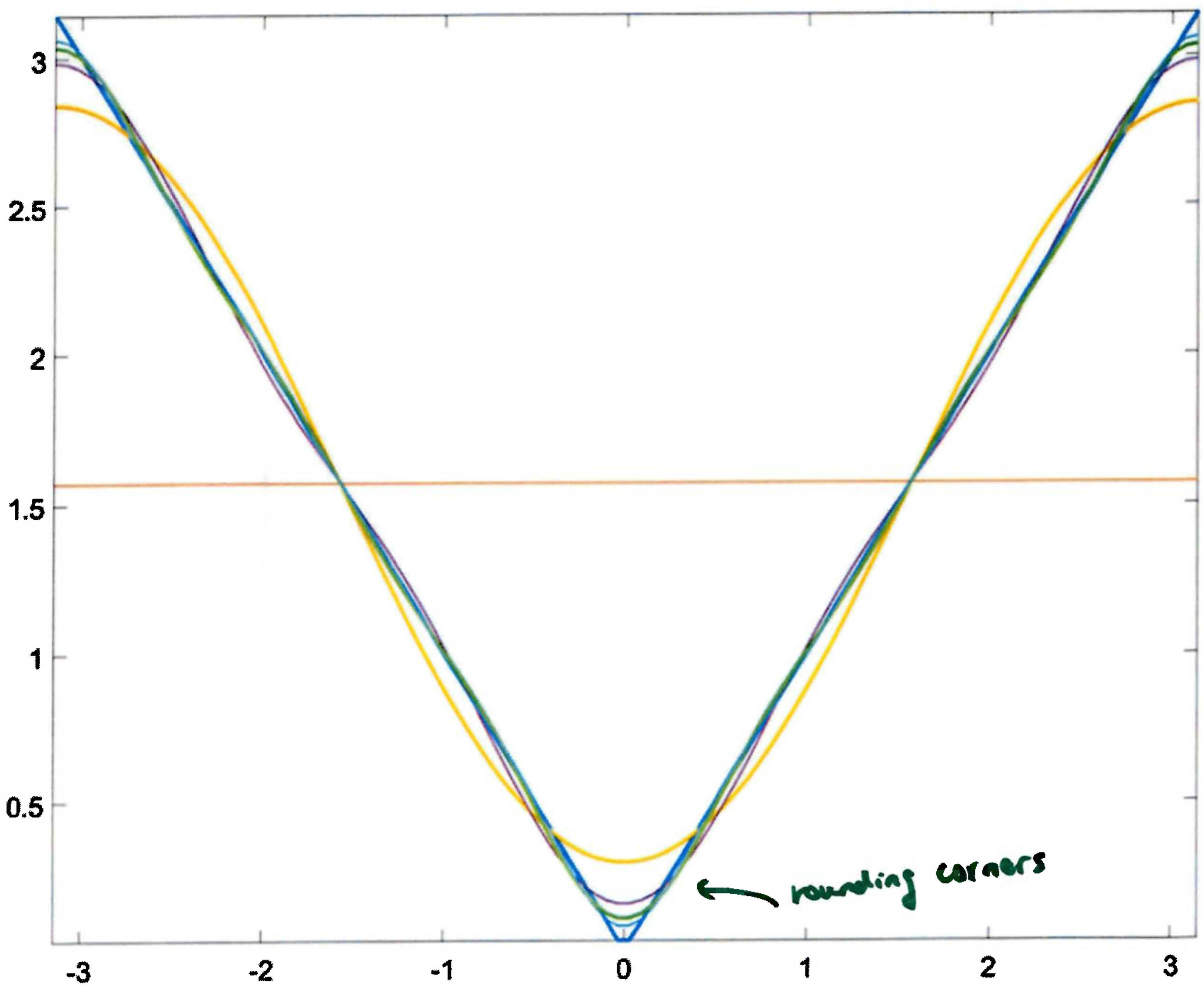


$$f(t) = \begin{cases} -t & -\pi \le t \le 0 \\ t & 0 \le t \le \pi \end{cases}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} -t\,dt + \int_{0}^{\pi} t\,dt\right] = \cdots = \pi$$

$$a_n = \cdots = \frac{2}{n^2\pi}\left((-1)^n - 1\right) \qquad b_n = \cdots = 0$$



3
2.5
2
1.5
1
0.5
-3
-2
-1
0
1
2
3
rounding corners