9.2 General Fourier Series and Convergence

Fourier series $\quad \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(nt) + b_n \sin(nt)\right]$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

$f(t)$ periodic with period 2π
defined on $-\pi < t < \pi$
} make more general

first, let's lift the period 2π part, make it $2L$

then function is defined on $-L < t < L$ $\quad$ L: half period

let's redefine "time" T

want $T = -L$ when $t = -\pi$
$T = L$ when $t = \pi$
} $T = \frac{L}{\pi} t$

so, $t = \frac{\pi}{L} T$ replace t in the formulas at the top

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt \qquad t = \frac{\pi}{L}T \quad \text{so } dt = \frac{\pi}{L}dT$$

$$= \frac{1}{\cancel{\pi}}\int_{-L}^{L} f(T)\cos\left(\frac{n\pi}{L}T\right)\cdot\frac{\cancel{\pi}}{L}\,dT = \frac{1}{L}\int_{-L}^{L} f(T)\cos\left(\frac{n\pi}{L}T\right)dT$$

repeat for b_n, we get

$$b_n = \frac{1}{L}\int_{-L}^{L} f(T)\sin\left(\frac{n\pi}{L}T\right)dT$$

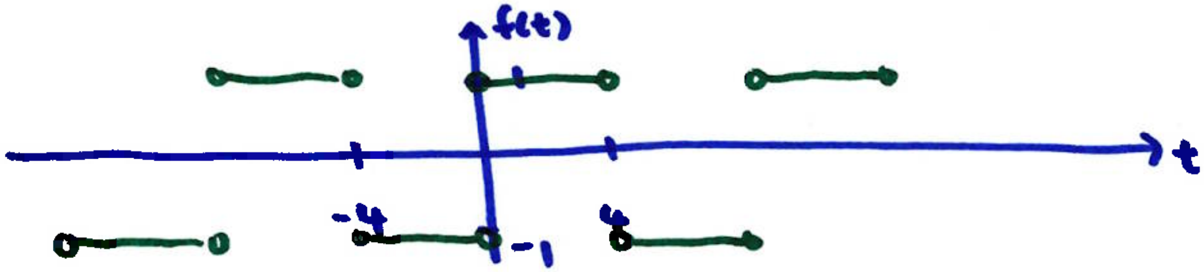
basis functions are now $\cos\left(\frac{n\pi}{L}T\right)$ and $\sin\left(\frac{n\pi}{L}T\right)$

(if $L=\pi$, back to original form)

T is just a variable, let's call it t

$$\boxed{\begin{array}{l} f(t) \text{ period } 2L \text{ defined on } -L < t < L \\ \text{Fourier series } \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi}{L}t\right) + b_n\sin\left(\frac{n\pi}{L}t\right)\right] \\ a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)dt \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi}{L}t\right)dt \end{array}}$$

<u>example</u> $f(t) = \begin{cases} -1 & -4 < t < 0 \\ 1 & 0 < t < 4 \end{cases}$ period 8 L is half period so $L = 4$



$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)dt \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi}{L}t\right)dt$$

$$a_0 = \frac{1}{4}\int_{-4}^{0} -1\cdot dt + \frac{1}{4}\int_{0}^{4} 1\cdot dt = 0 \quad \text{(area under curve)}$$

$$a_n = \frac{1}{4}\int_{-4}^{0} -1\cdot\cos\left(\frac{n\pi}{4}t\right)dt + \frac{1}{4}\int_{0}^{4} 1\cdot\cos\left(\frac{n\pi}{4}t\right)dt$$

$$= -\frac{1}{4}\left(\frac{4}{n\pi}\sin\left(\frac{n\pi}{4}t\right)\Big|_{-4}^{0}\right) + \frac{1}{4}\left(\frac{4}{n\pi}\sin\left(\frac{n\pi}{4}t\right)\Big|_{0}^{4}\right)$$

$= 0 + 0 = 0$ no cosine terms in Fourier series ~~except a_0~~

$$b_n = \frac{1}{4}\int_{-4}^{0} -1 \cdot \sin\left(\frac{n\pi}{4}t\right)dt + \frac{1}{4}\int_{0}^{4} 1 \cdot \sin\left(\frac{n\pi}{4}t\right)dt$$

$$= \frac{1}{4}\left(\frac{4}{n\pi}\cos\left(\frac{n\pi}{4}t\right)\right|_{-4}^{0} - \frac{1}{4}\left(\frac{4}{n\pi}\cos\left(\frac{n\pi}{4}t\right)\right|_{0}^{4}$$

$$= \frac{1}{4}\left(\frac{4}{n\pi} - \frac{4}{n\pi}\underbrace{\cos(-n\pi)}_{(-1)^n}\right) - \frac{1}{4}\left(\frac{4}{n\pi}\underbrace{\cos(n\pi)}_{(-1)^n} - \frac{4}{n\pi}\right)$$

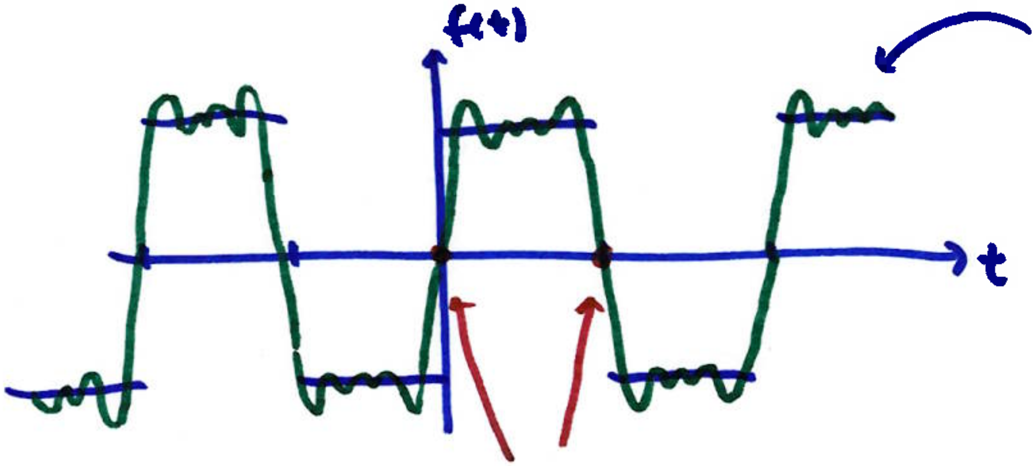
$$= \frac{2}{n\pi} - \frac{2}{n\pi}(-1)^n = \frac{2}{n\pi}\left[1-(-1)^n\right] \qquad n = 1, 2, 3, \ldots$$

$$= \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

Fourier series representation

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right]\sin\left(\frac{n\pi}{4}t\right) = \sum_{n=\text{odd}}^{\infty} \frac{4}{n\pi}\sin\left(\frac{n\pi}{4}t\right)$$

$$= \frac{4}{\pi}\sin\left(\frac{\pi}{4}t\right) + \frac{4}{3\pi}\sin\left(\frac{3\pi}{4}t\right) + \frac{4}{5\pi}\sin\left(\frac{5\pi}{4}t\right) + \cdots$$

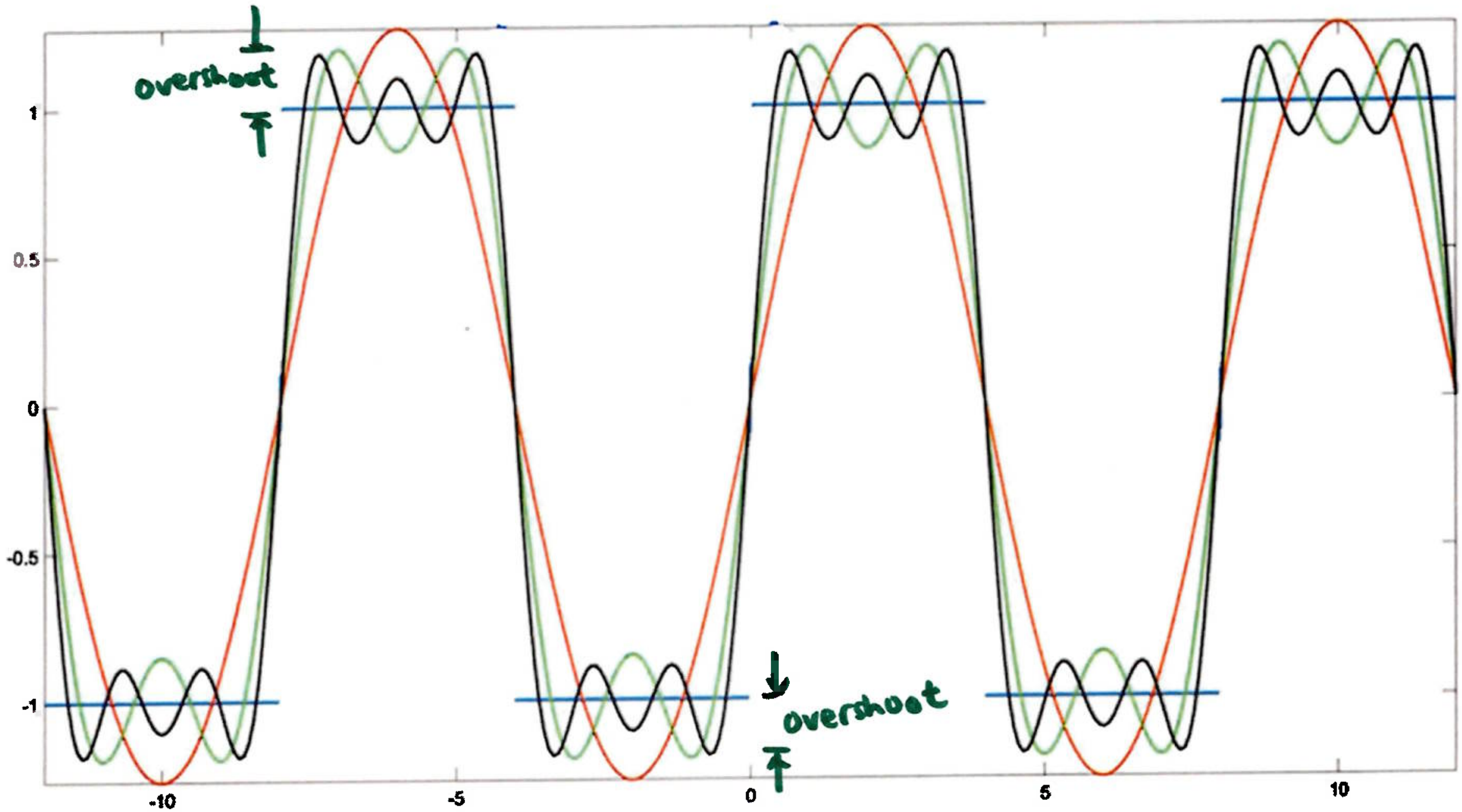


Fourier series converges to f(t) wherever f(t) is continuous

the more terms, the better the series tracks f(t)

Fourier series converges to the average of the first point on the left and the first point on the right

at discontinuity $\dfrac{f(t^-)+f(t^+)}{2}$

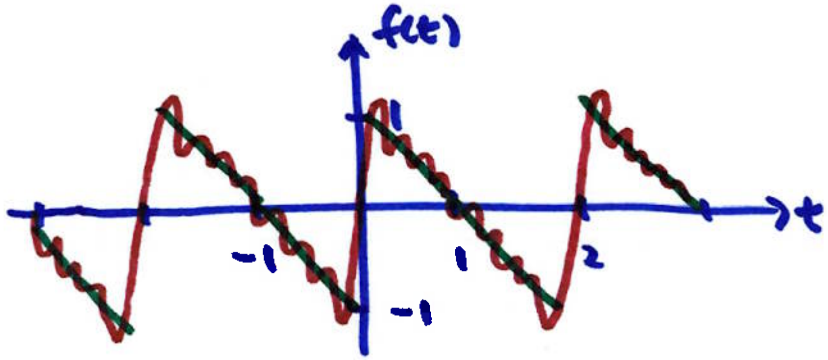


that overshoot is always about 9 % of the jump
(here, jump is 2 , overshoot $\approx$ 0.18) regardless of order (n)

Gibbs Phenomenon

<u>example</u>

$$f(t) = \begin{cases} -1-t & -1<t<0 \\ 1-t & 0<t<1 \end{cases} \qquad \text{period } 2 \qquad L=1$$



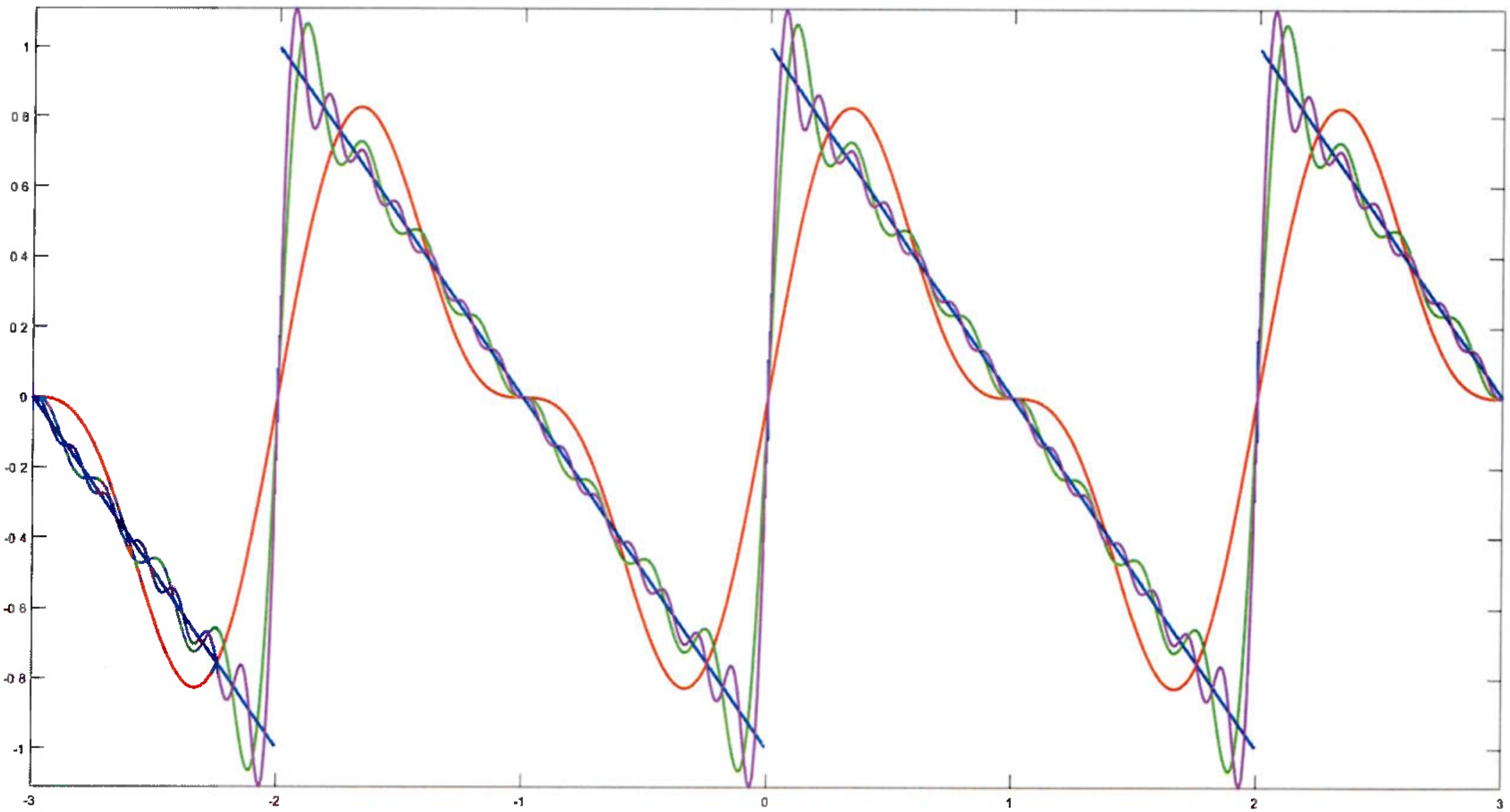
$$a_0 = \cdots = 0$$

$$a_n = \cdots = 0$$

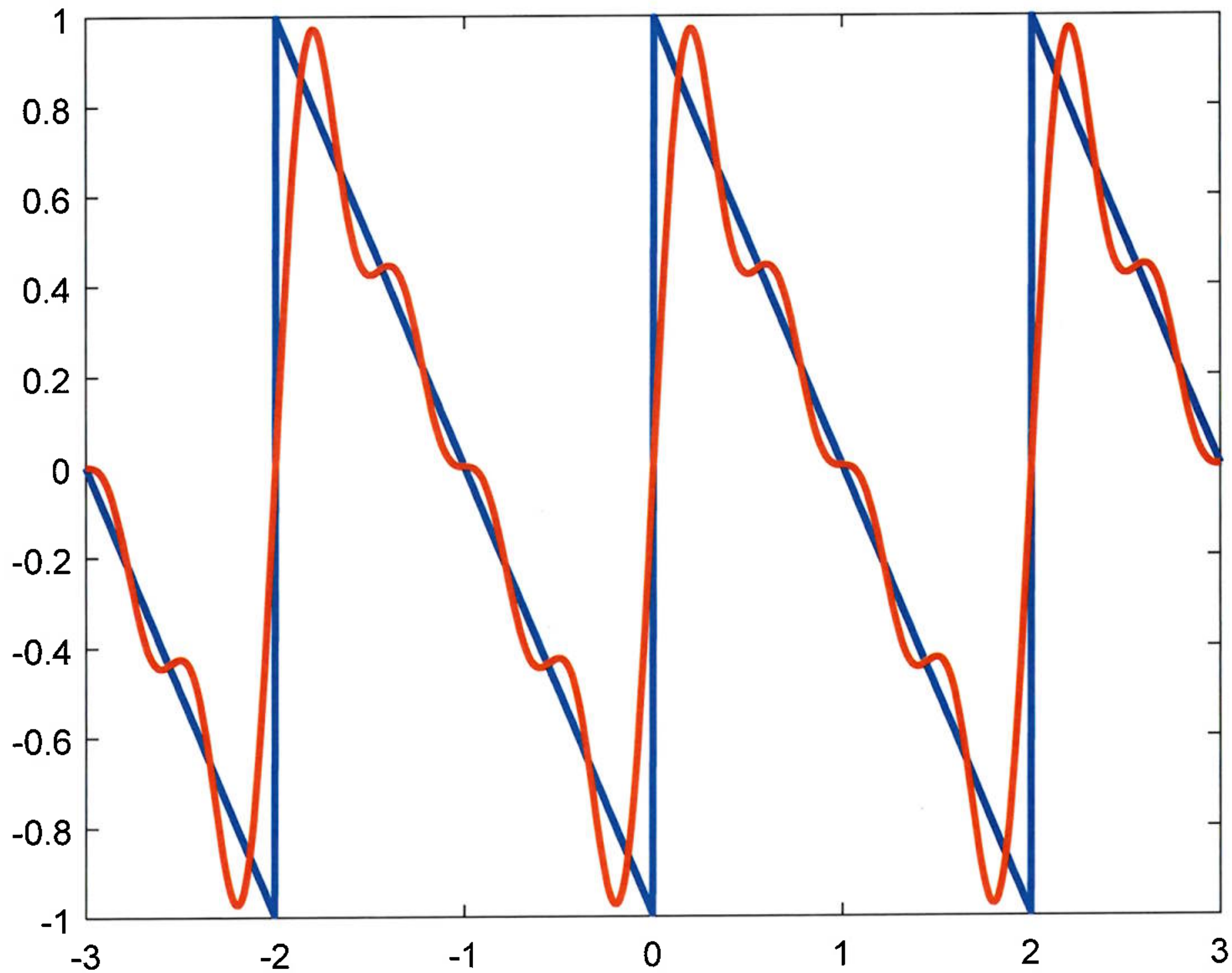
$$b_n = \cdots = \frac{2}{n\pi}$$

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin\left(\frac{n\pi}{1} t\right)$$

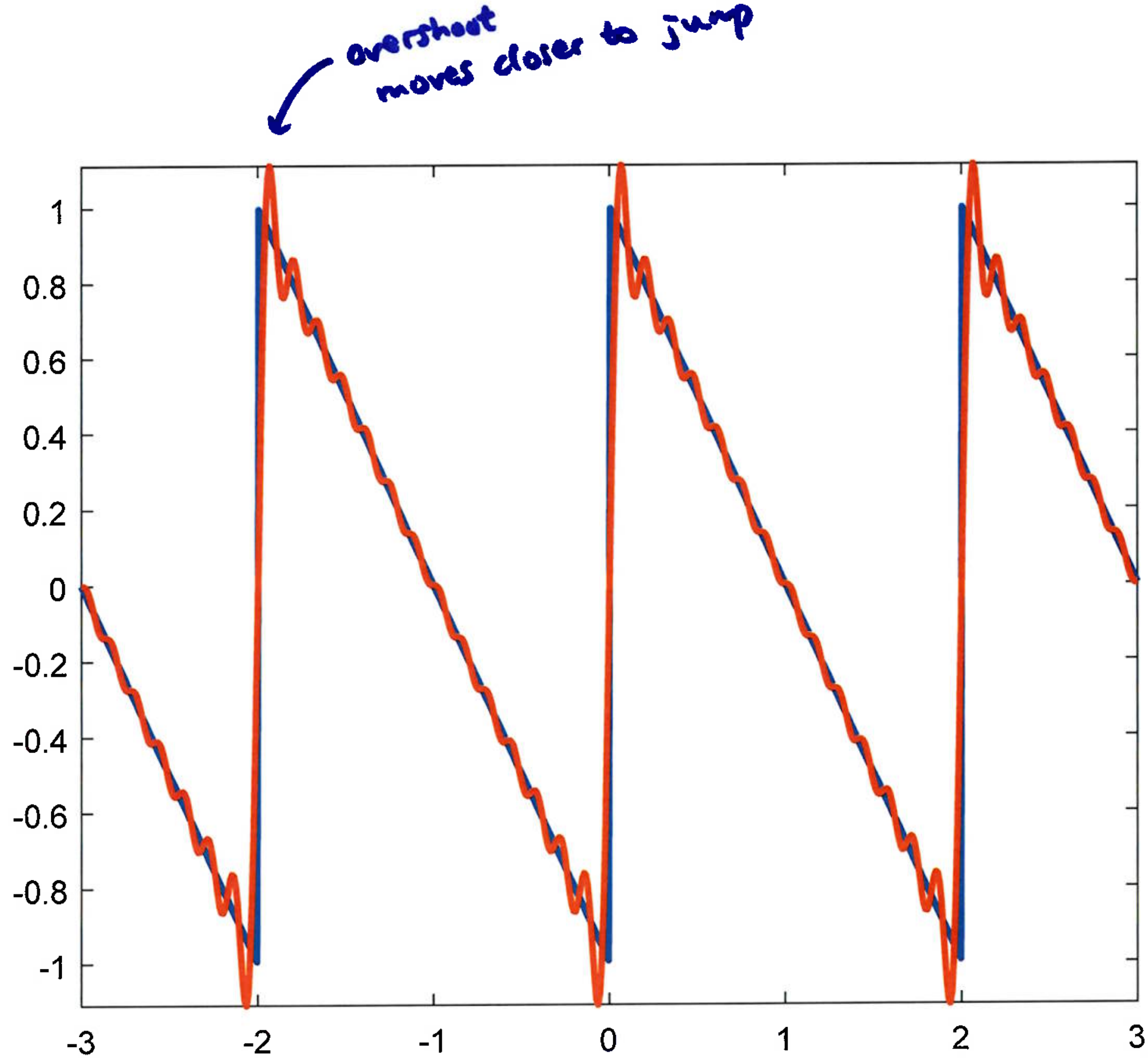
Overshoot
moves depending on n
but, again, ~9% of jump



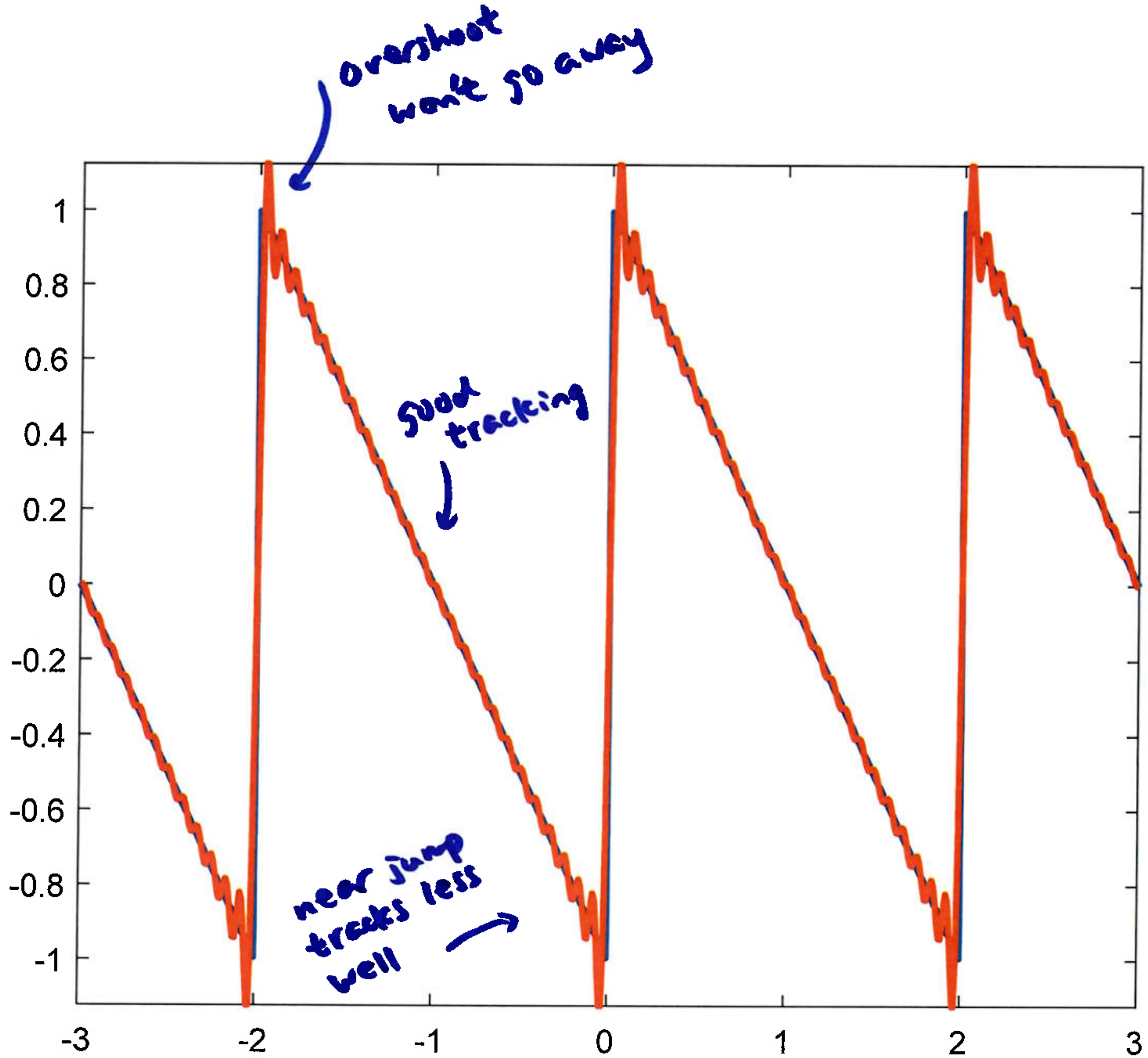
n = 5



n = 15


overshoot moves closer to jump
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-3
-2
-1
0
1
2
3

n = 25


Overshoot won't go away
good tracking
near jump tracks less well
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-3
-2
-1
0
1
2
3