9.2 General Fourier Series (part 2)

$f(t)$ period $2L$ defined on $-L < t < L$

its Fourier series is $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}t\right)$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)dt$$

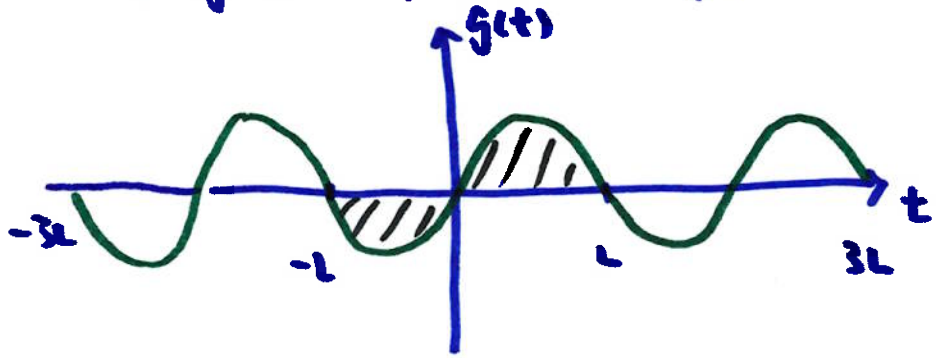
$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi}{L}t\right)dt$$

in practice, we normally start t at 0 not $-L$

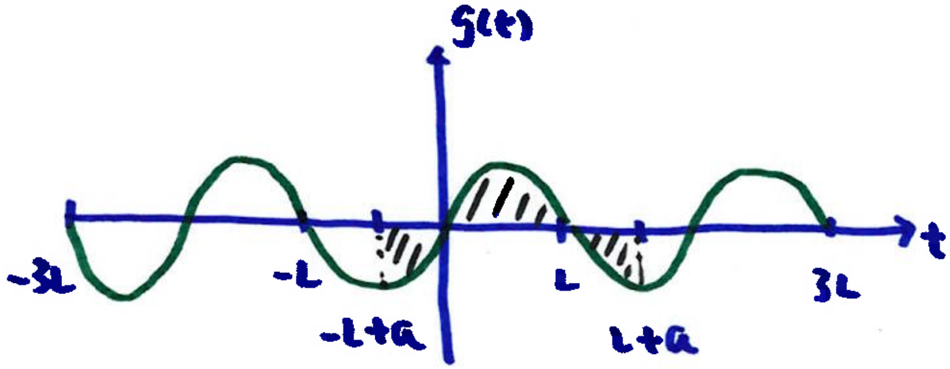
$f(t)$ is still periodic w/ period $2L$, but now $0 < t < 2L$

how do the formulas change?

if $g(t)$ is periodic w/ period $2L$



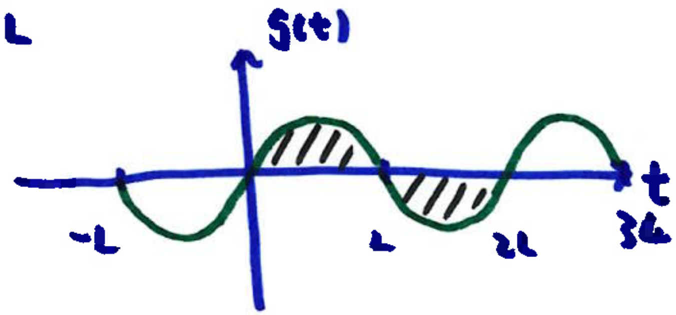
$\int_{-L}^{L} g(t)\,dt$ is the area of the shaded region



$\int_{-L+a}^{L+a} g(t)\,dt$ is the area of shaded region

same area as $\int_{-L}^{L} g(t)\,dt$

if $a = L$



clearly the same area as

$$\int_{-L}^{L} g(t)\,dt$$

as long as $g(t)$ has period $2L$ and the interval of integration is also $2L$, $\int_{-L+a}^{L+a} g(t)\,dt$ is the same for any a

revisit $$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)dt$$

period $2L$

period $\frac{2\pi}{\frac{n\pi}{L}} = \frac{2L}{n}$ → but multiple is still period

So, $2L$ is also a period

So, $f(t)\cos\left(\frac{n\pi}{L}t\right)$ also ~~is~~ has period $2L$

now we can shift the interval however we want, as long as it is $2L$

more useful formula for Fourier series $f(t)$ period $2L$

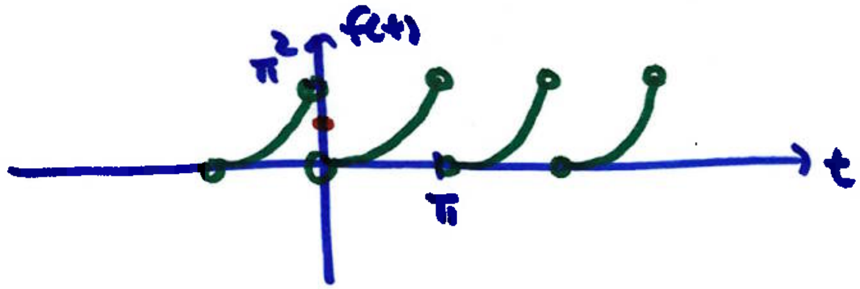
$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}t\right)$$

$$a_n = \frac{1}{L}\int_0^{2L} f(t)\cos\left(\frac{n\pi}{L}t\right)dt$$

$$b_n = \frac{1}{L}\int_0^{2L} f(t)\sin\left(\frac{n\pi}{L}t\right)dt$$

or *any* interval length $2L$

<u>example</u> $f(t) = t^2$ for $0 < t < \pi$ period π



L : half period

here, $L = \frac{\pi}{2}$

$$a_n = \frac{1}{L}\int_0^{2L} f(t)\cos\left(\frac{n\pi}{L}t\right)dt$$

$$= \frac{2}{\pi}\int_0^{\pi} t^2 \cos(2nt)\,dt$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} t^2\,dt = \cdots = \frac{2}{3}\pi^2$$

$$a_n = \cdots = \frac{2}{\pi}\left[\frac{(2n^2\pi - 1)\overbrace{\sin(2n\pi)}^{0} + 2n\pi\overbrace{\cos(2n\pi)}^{1}}{4n^3}\right]$$

$$= \cdots = \frac{1}{n^2}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} t^2 \sin(2nt)\,dt = \cdots = -\frac{\pi}{n}$$

Fourier series is

$$F(t) = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}\left[\frac{1}{n^2}\cos(2nt) - \frac{\pi}{n}\sin(2nt)\right]$$

$F(t)$ converges to $f(t)$ where $f(t)$ is defined

$$F\left(\frac{\pi}{2}\right) \to f\left(\frac{\pi}{2}\right)$$

$F(t)$ converges to $\frac{1}{2}\left[f(t^-) + f(t^+)\right]$ where $f(t)$ is discontinuous

for this example,

$$F(0) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{2}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{2} - \frac{\pi^2}{3}$$

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}}$$

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots = \frac{\pi^2}{6}$$

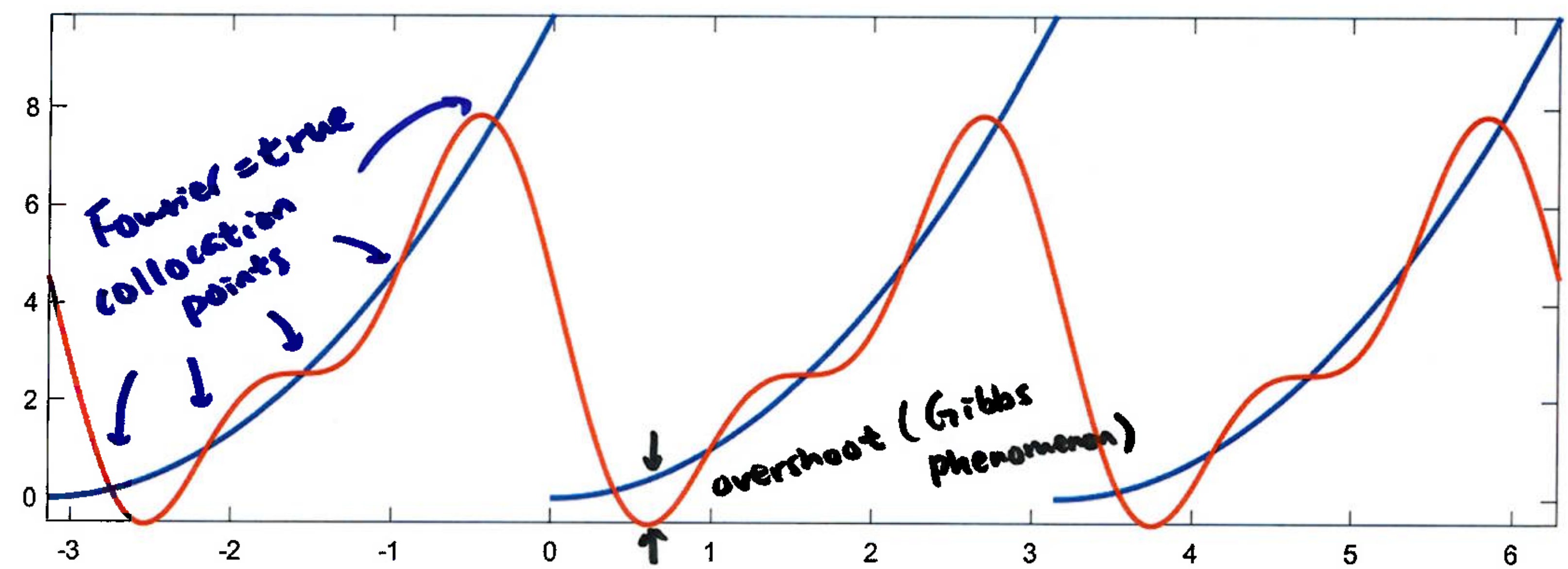
"Basel Problem"

↓

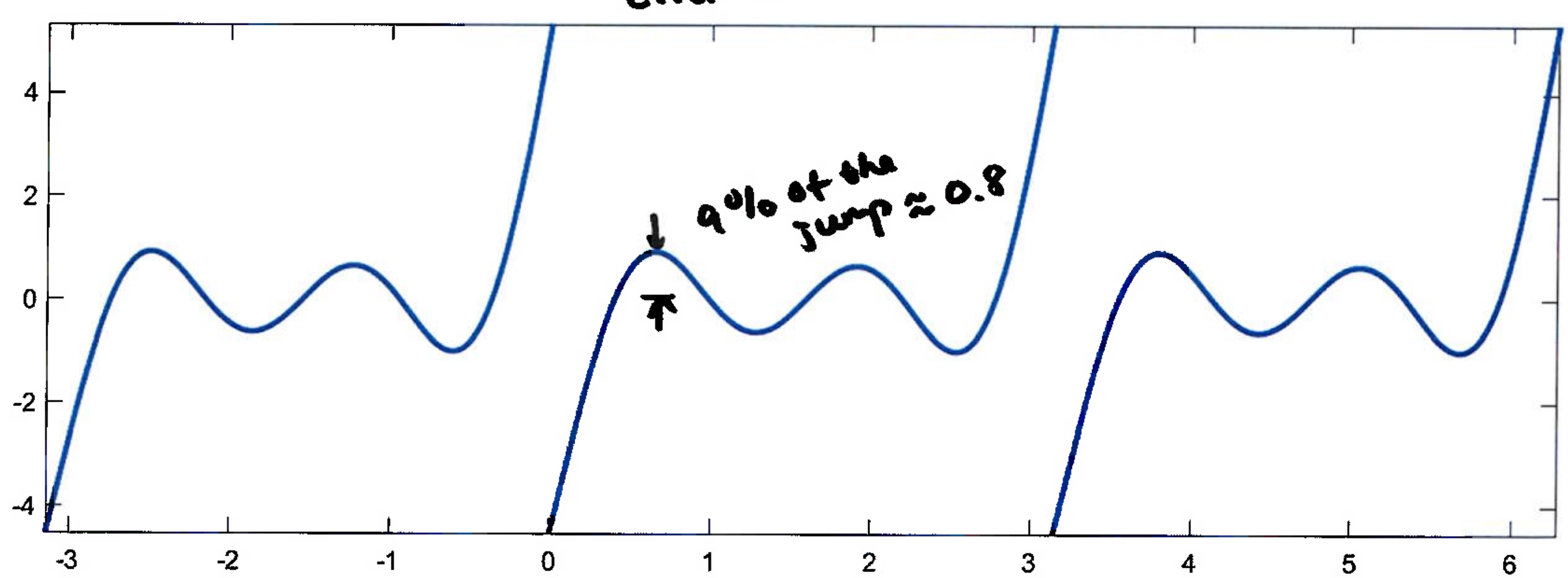
Hometown of

Euler + Bernoulli

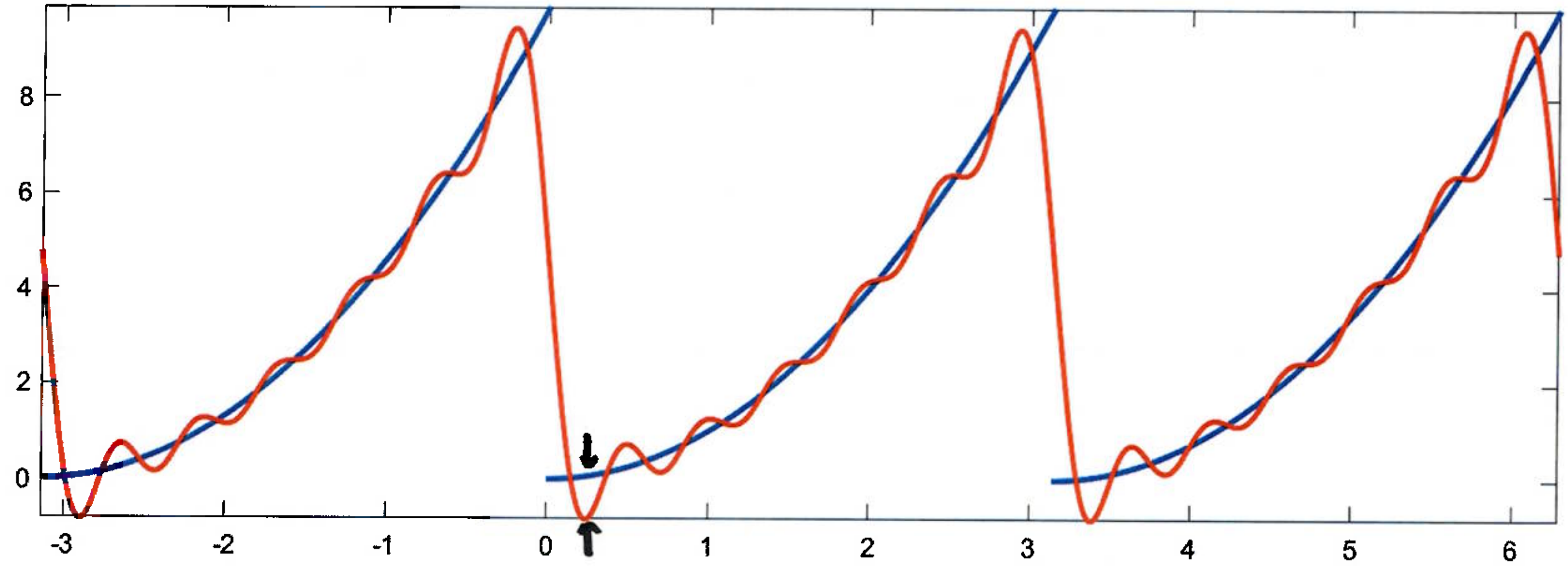
n=3

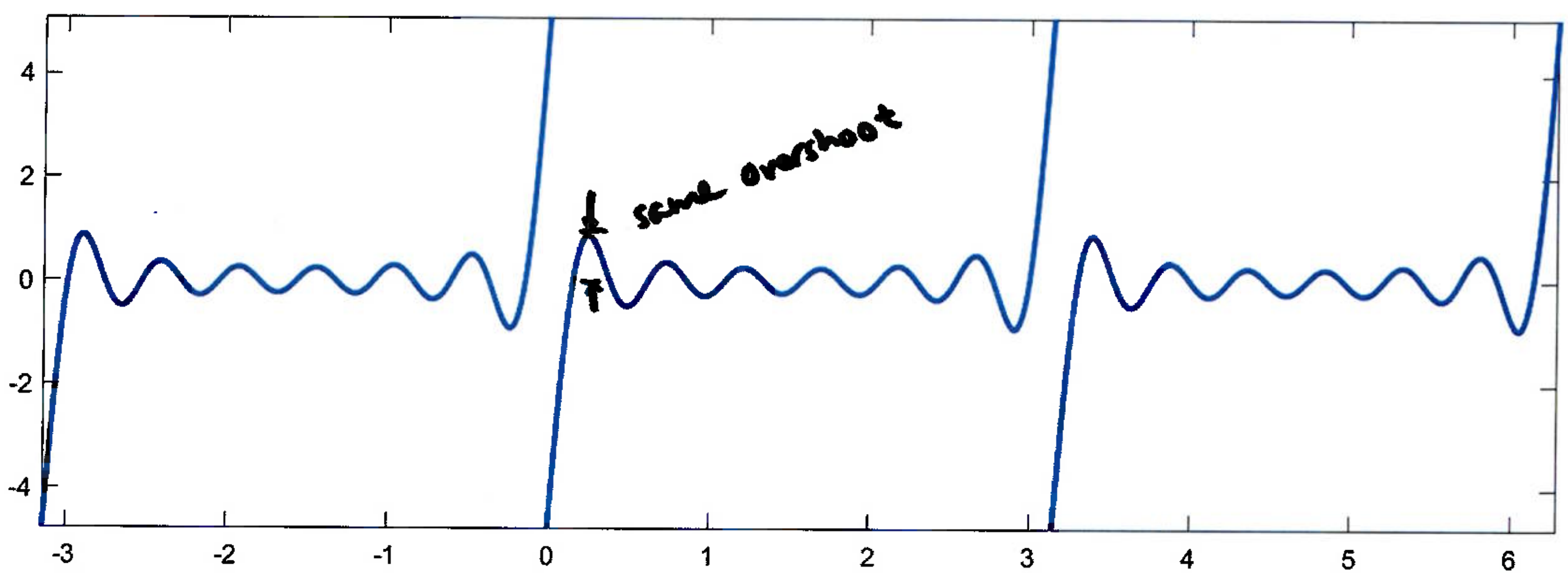


Error (Fourier - true)

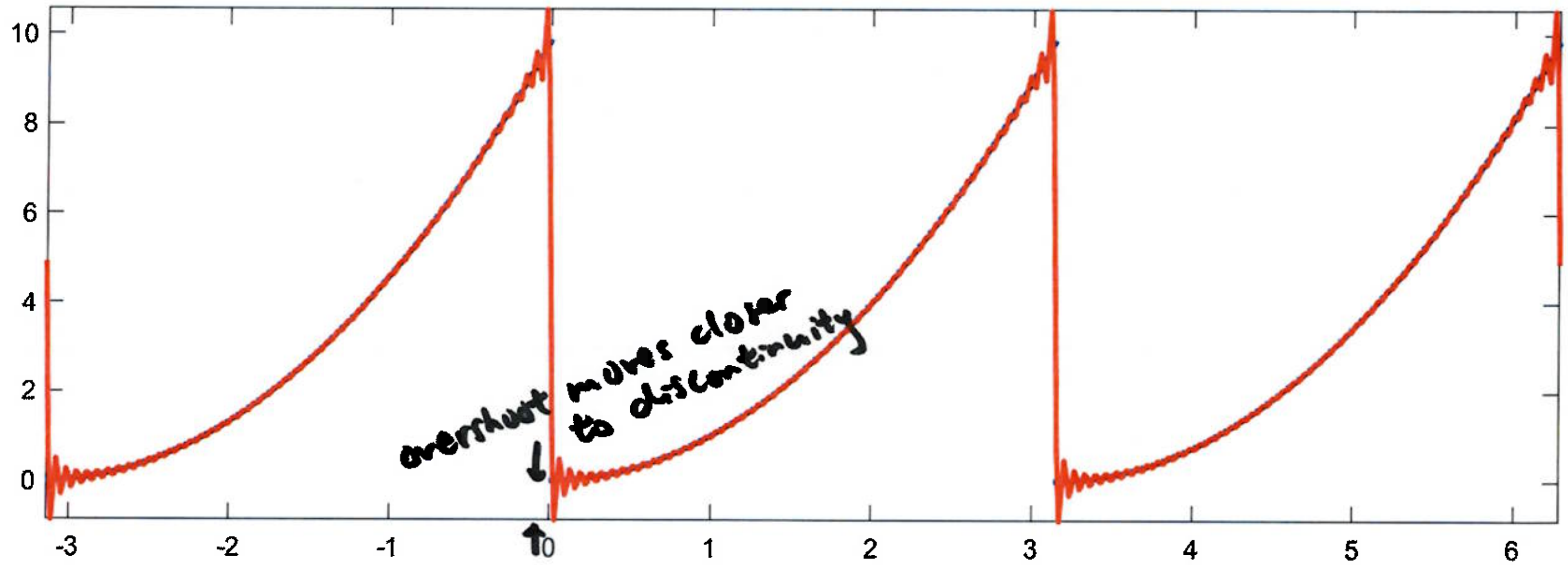


n=7

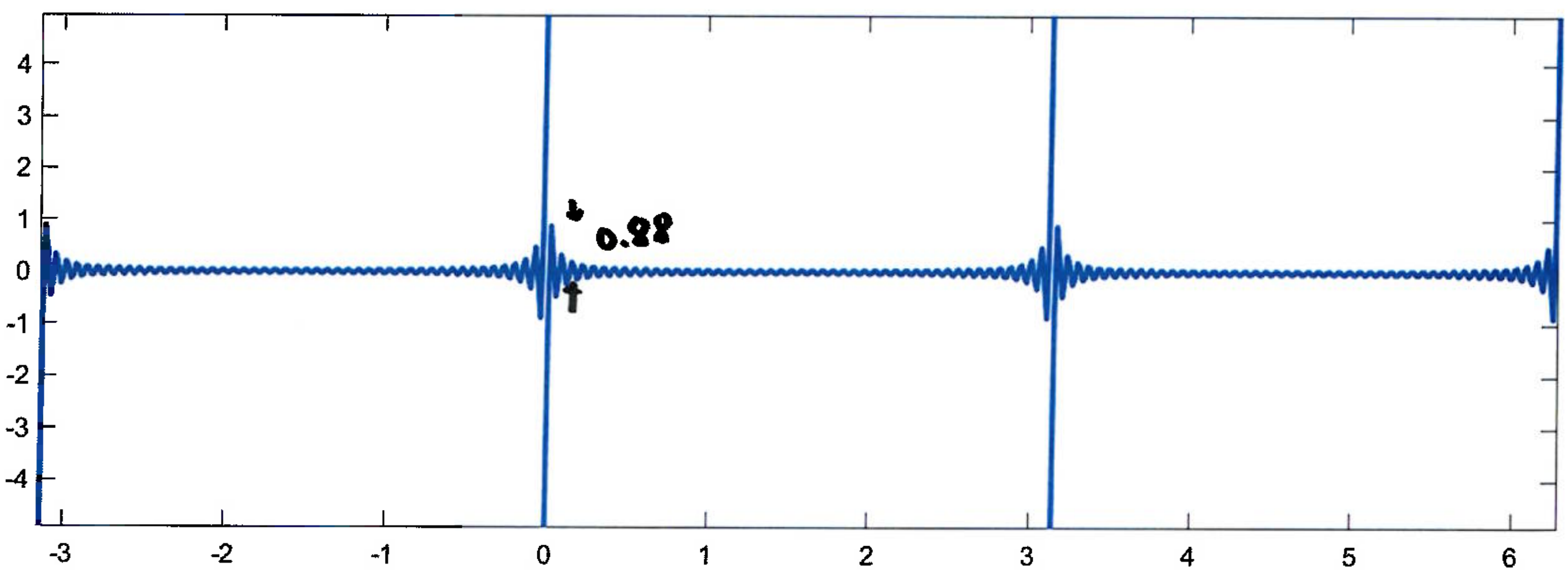




n= 49



10
8
6
4
2
0
-3
-2
-1
0
1
2
3
4
5
6
overshoot moves closer to discontinuity



4
3
2
1
0
-1
-2
-3
-4
-3
-2
-1
0
1
2
3
4
5
6
0.88