9.3 Fourier Cosine and Sine Series (part 2)

$f(t)$ period $2L$

Cosine series (even extensions) $b_n = 0$, $a_n = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{n\pi}{L}t\right)dt$

Sine series (odd extensions) $a_n = 0$, $b_n = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi}{L}t\right)dt$

let's apply this to solving a boundary-value problem

for example, $x'' + 4x = 0$ $\quad \underbrace{x(0) = 0, \; x(\pi) = 0}_{\text{conditions at the boundary}}$ positions specified

Dirichlet condition

or $\quad x'' + 4x = 0$ $\quad \overbrace{x'(0) = 0, \; x'(\pi) = 0}$ Neumann condition

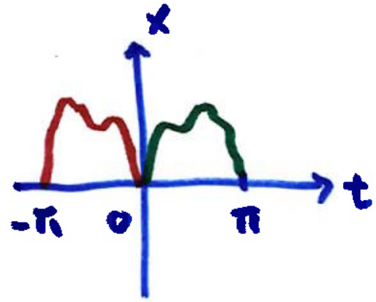
NOT like an initial-value prob

$x(0) = x_0, \; x'(0) = x_0'$

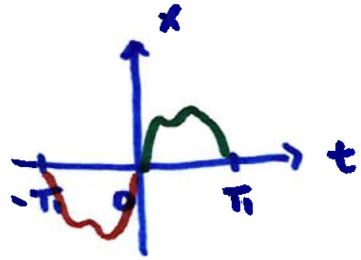
we can interpret this problem as finding a periodic $x(t)$

with half-period $L = \pi$

we can choose to add even or odd extensions



w/ even extension → cosine series



w/ odd extension → sine series

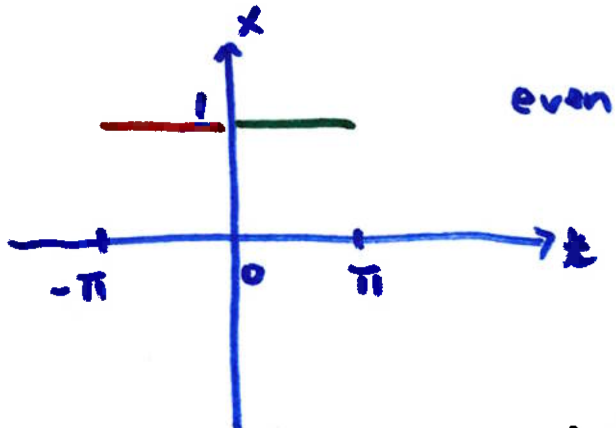
if homogeneous (right side zero), either is fine

if right side is not zero, then that function, along with boundary conditions, determine the type of expansion (cosine, sine, or full) example $\quad x'' + 2x = 1 \qquad x(0) = 0 \quad x(\pi) = 0$ (homogeneous BC's both zero)

first, we need to decide whether to expand $x(t)$ as cosine or sine series

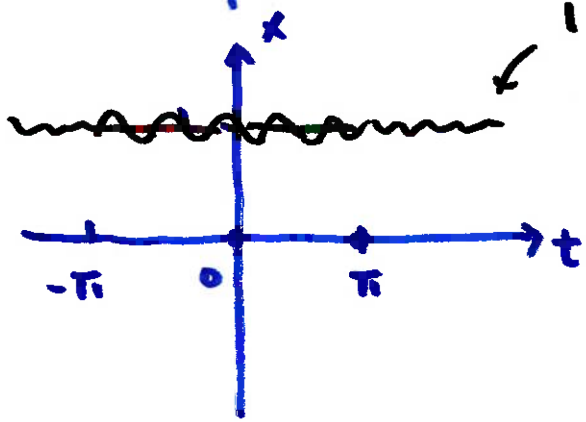
→ determined by whether 1 (right side) should be a cosine or sine series

Key: that expansion MUST also meet the BC's $x(0)=0, \ x(\pi)=0$

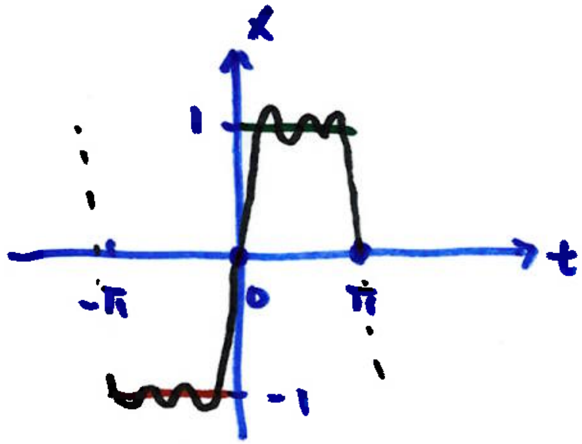


even extension (cosine series)

1 as cosine series



here, 1 as a cosine series CANNOT satisfy the BC's



odd extension (sine series)

this now meets the BC's

so, we choose sine series

(if the BC's were $x'(0) = x'(\pi) = 0$, cosine series works but sine doesn't)

$$x'' + 2x = 1 \qquad x(0) = x(\pi) = 0$$

expand 1 as a sine series $\quad 1 \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right]\sin(nt)$

x is also a sine series but with unknown coefficients w/ the same L as right side

$$x = \sum_{n=1}^{\infty} B_n \sin(nt)$$

→ must satisfy the diff. eq. plug it in.

$$= B_1 \sin(t) + B_2 \sin(2t) + B_3 \sin(3t) + \cdots$$

$$x' = \sum_{n=1}^{\infty} n B_n \cos(nt)$$

$$x'' = \sum_{n=1}^{\infty} -n^2 B_n \sin(nt)$$

$x'' + 2x = 1$ becomes

$$\sum_{n=1}^{\infty} -n^2 B_n \sin(nt) + 2\sum_{n=1}^{\infty} B_n \sin(nt) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right] \sin(nt)$$

for each n, the coefficients of $\sin(nt)$ must match

$$-n^2 B_n + 2B_n = \frac{2}{n\pi}\left[1-(-1)^n\right] \qquad n = 1, 2, 3, \dots$$

$$B_n = \frac{2\left[1-(-1)^n\right]}{n\pi(2-n^2)} \qquad n \neq 0,\ 2-n^2 \neq 0 \text{ for } n = 1, 2, 3, \dots$$

solution:

$$\boxed{x(t) = \sum_{n=1}^{\infty} \frac{2\left[1-(-1)^n\right]}{n\pi(2-n^2)} \sin(nt)} = \frac{4}{\pi}\sin(t) - \frac{4}{21\pi}\sin(3t) - \frac{4}{105\pi}\sin(5t) - \dots$$

coefficient decreasing rapidly for high n

→ fast converging series

only few terms needed

$2-n^2$ in denominator is not a problem here

but if $x'' + 4x = 1$ $\quad x(0) = x(\pi) = 0$

$2-n^2$ becomes $4-n^2$ → problem when $n=2$

→ resonance