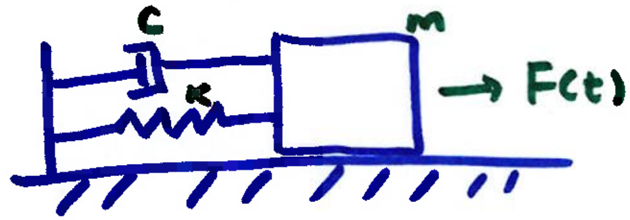
9.4 Application of Fourier Series

mass-spring-damper



m: mass k: spring constant F(t): external force

c: damping constant

x(t): displacement from equilibrium

$$mx'' + cx' + kx = F(t)$$

let's look at the case with $c=0$, $F(t)$ periodic

$$mx'' + kx = F(t)$$

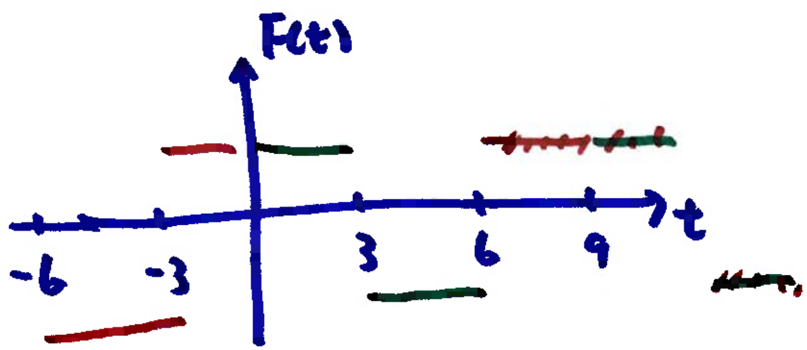
general solution: $x(t) = \underbrace{c_1 \cos\left(\sqrt{\frac{k}{m}}\,t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\,t\right)}_{\text{complementary solution (right side 0)}} + \underbrace{x_{particular}}_{\text{due to } F(t)}$

let's focus on $x_{particular} = x_p$ if $F(t)$ is periodic

<u>example</u> $m = 1$, $k = 5$, $F(t) = \begin{cases} 1 & 0 < t < 3 \\ -1 & 3 < t < 6 \end{cases}$ period 6

first, write $F(t)$ as a Fourier cosine or sine series

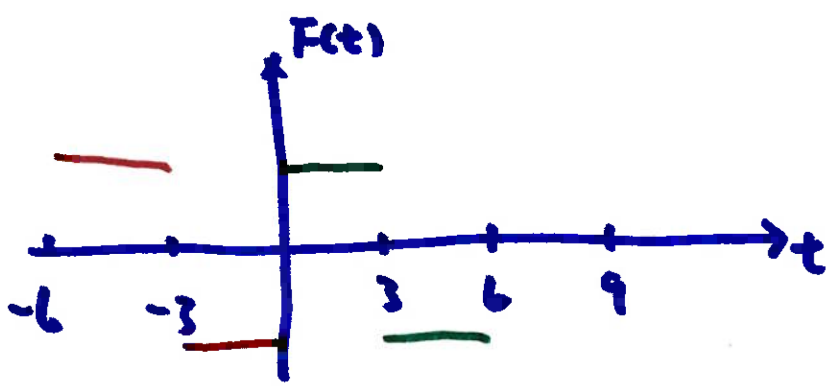
→ want symmetry with respect to $t = 0$



as cosine series

($F(t)$ is even)

not symmetric in the sense that the previous 6 seconds the force is $-F(t)$



as sine series

($F(t)$ odd)

is symmetric because every 6 sec interval is the same as $F(t)$

so, choose sine series

$$F(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right]\sin\left(\frac{n\pi t}{3}\right) \qquad (L=3)$$

$$x'' + 5x = F(t) = \frac{4}{\pi}\sin\left(\frac{\pi t}{3}\right) + \frac{4}{3\pi}\sin\left(\frac{3\pi t}{3}\right) + \frac{4}{5\pi}\sin\left(\frac{5\pi t}{3}\right) + \cdots$$

for each n, we solve $x'' + 5x = \frac{2}{n\pi}\left[1-(-1)^n\right]\sin\left(\frac{n\pi t}{3}\right)$

means particular solution
in the form of
$A_n\cos\left(\frac{n\pi t}{3}\right) + B_n\sin\left(\frac{n\pi t}{3}\right)$
(from undetermined coeffs)

combining ALL possible n's, our particular solution is

$$x_p = \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi t}{3}\right) + B_n\sin\left(\frac{n\pi t}{3}\right)$$

goal: find A_n, B_n

must satisfy $x'' + 5x = F(t)$, so plug x_p into that

$$x_p' = \sum_{n=1}^{\infty} -\frac{n\pi}{3}A_n\sin\left(\frac{n\pi t}{3}\right) + \frac{n\pi}{3}B_n\cos\left(\frac{n\pi t}{3}\right)$$

$$x_p'' = \sum_{n=1}^{\infty} -\frac{n^2\pi^2}{3^2}A_n\cos\left(\frac{n\pi t}{3}\right) - \frac{n^2\pi^2}{3^2}B_n\sin\left(\frac{n\pi t}{3}\right)$$

into $x'' + 5x = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right] \sin\left(\frac{n\pi t}{3}\right)$

$$\sum_{n=1}^{\infty} -\frac{n^2\pi^2}{3^2} A_n \cos\left(\frac{n\pi t}{3}\right) - \frac{n^2\pi^2}{3^2} B_n \sin\left(\frac{n\pi t}{3}\right)$$

$$+ 5 \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi t}{3}\right) + B_n \sin\left(\frac{n\pi t}{3}\right) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right] \sin\left(\frac{n\pi t}{3}\right)$$

equate coefficients of $\cos\left(\frac{n\pi t}{3}\right)$ and $\sin\left(\frac{n\pi t}{3}\right)$

for each n

$\cos\left(\frac{n\pi t}{3}\right)$: $\quad -\frac{n^2\pi^2}{3^2} A_n + 5A_n = 0 \quad$ no $\cos\left(\frac{n\pi t}{3}\right)$ in $F(t)$

$$\rightarrow \boxed{A_n = 0}$$

$\sin\left(\frac{n\pi t}{3}\right)$: $\quad -\frac{n^2\pi^2}{3^2} B_n + 5B_n = \frac{2}{n\pi}\left[1-(-1)^n\right]$

$$\rightarrow \boxed{B_n = \frac{2\left[1-(-1)^n\right]}{n\pi\left(5 - \frac{n^2\pi^2}{3^2}\right)} = \frac{18\left[1-(-1)^n\right]}{n\pi\left(45 - n^2\pi^2\right)}}$$

"formal series" (convergence not verified)

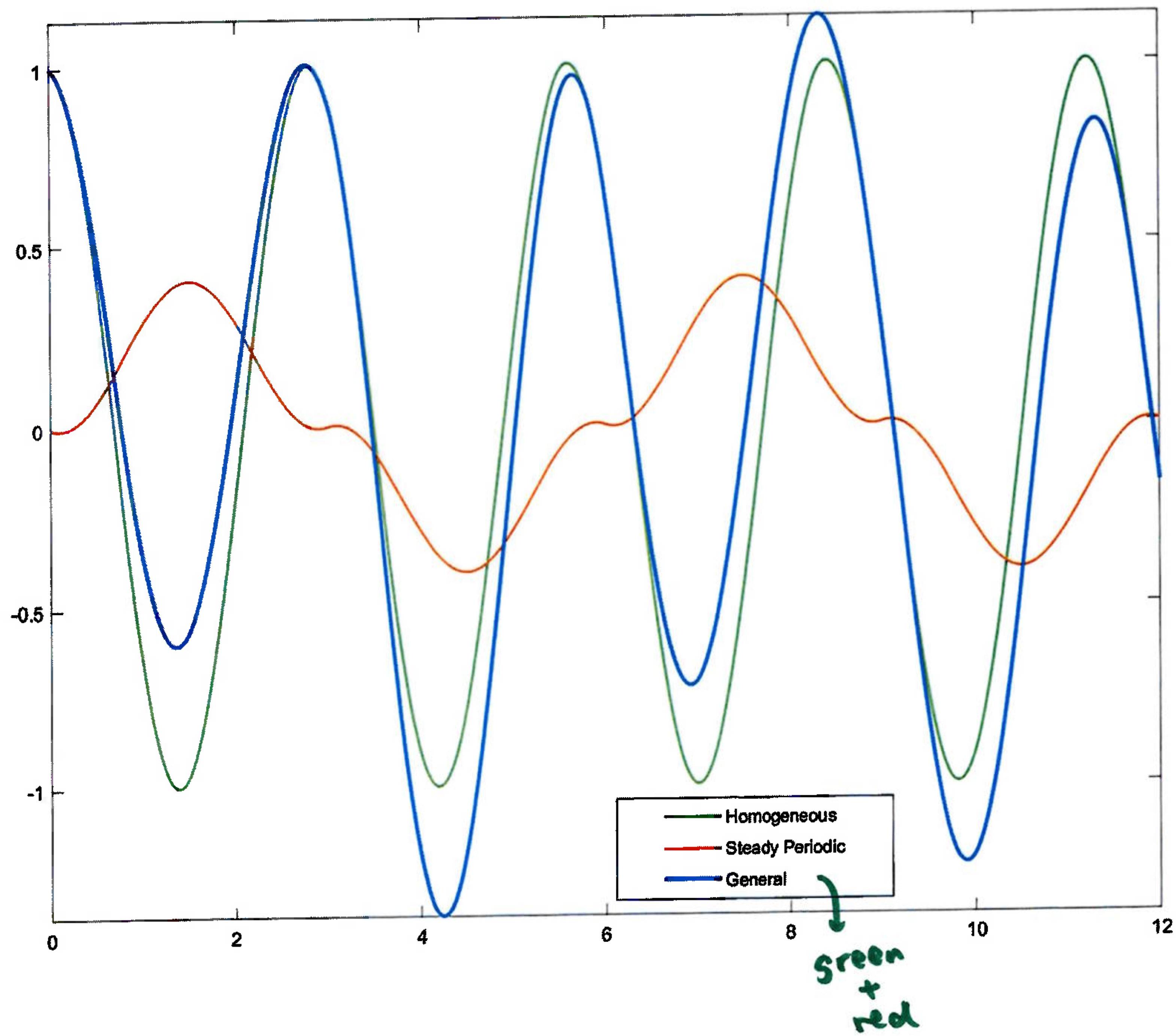
$$\boxed{x_p = \sum_{n=1}^{\infty} \frac{18\left[1-(-1)^n\right]}{n\pi\left(45-n^2\pi^2\right)} \sin\left(\frac{n\pi t}{3}\right)}$$

particular solution due to $F(t)$

↳ also called "steady periodic solution"

general: $x = C_1 \cos(\sqrt{5}\,t) + C_2 \sin(\sqrt{5}\,t)$ $\longrightarrow + x_p$

$x(0)=0 \quad x'(0)=1$


1
0.5
0
-0.5
-1
0
2
4
6
8
10
12
Homogeneous
Steady Periodic
General
green + red

look at the $45 - n^2\pi^2$ in the denominator of x_p

if it is exactly zero for some n, trouble (magnitude $\to \infty$)

applied force has frequency = fundamental frequency

for example, $x'' + 5x = \sin(\sqrt{5}t)$

$$x(t) = C_1\cos(\sqrt{5}t) + C_2\sin(\sqrt{5}t)$$
$$- \frac{1}{2\sqrt{5}}t\cos(\sqrt{5}t)$$

pure resonance

in our example, we are safe because no n ($n = 1, 2, 3, \ldots$)

can make $45 - n^2\pi^2 = 0$ exactly

no resonance in our example

but let's tweak the example a bit

<u>example</u> $x'' + 5x = F(t)$ $F(t) = \begin{cases} 1 & 0<t<10 \\ -1 & 10<t<20 \end{cases}$ period 20

same procedure

$$F(t) = \cdots = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right]\sin\left(\frac{n\pi t}{10}\right)$$

again, following the same steps,

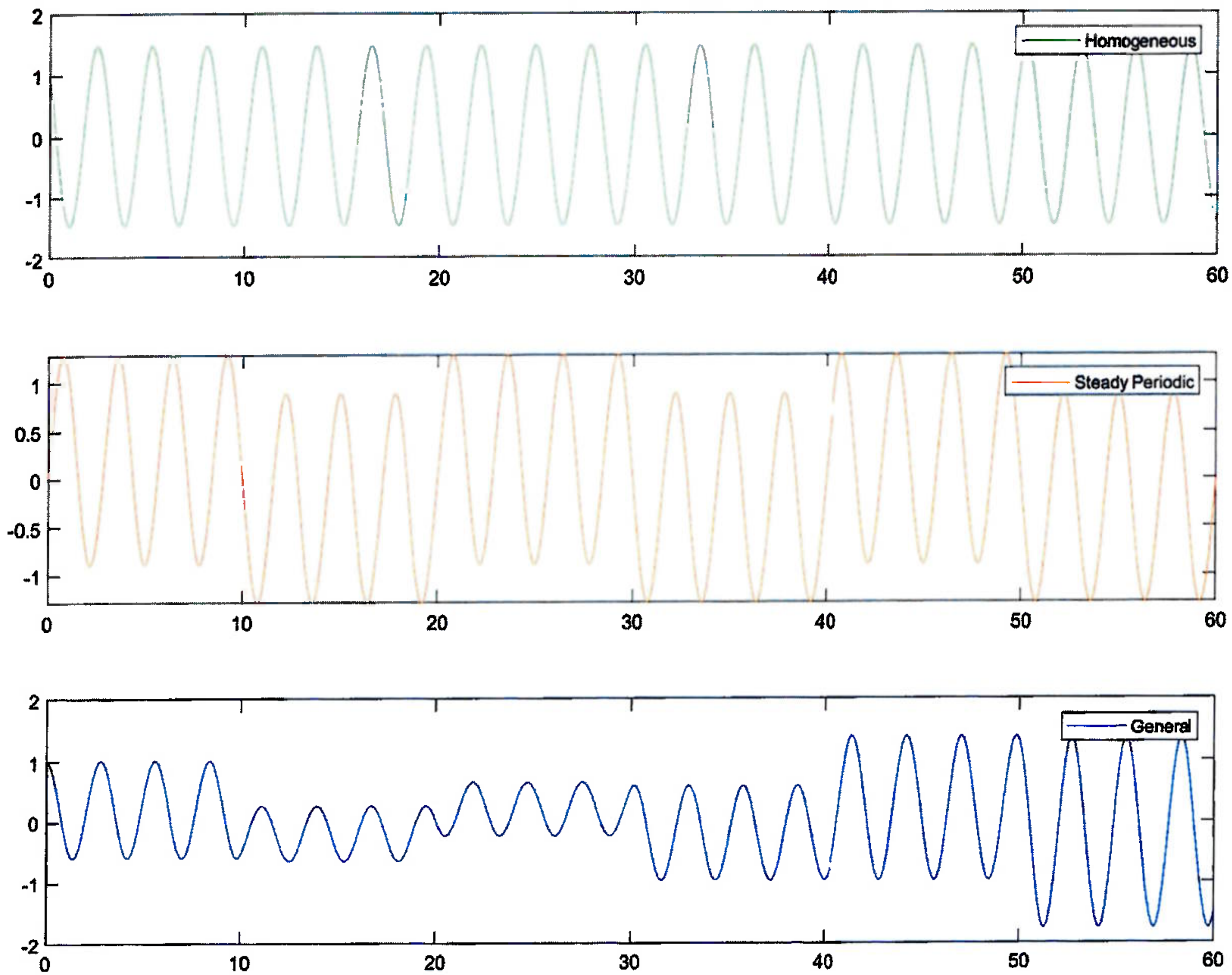
⋮

$$x_p = \sum_{n=1}^{\infty} \frac{200\left[1-(-1)^n\right]}{n\pi\left(500-n^2\pi^2\right)} \sin\left(\frac{n\pi t}{10}\right)$$

$$= 0.26\sin\left(\frac{\pi}{10}t\right) + 0.1\sin\left(\frac{3\pi}{10}t\right) + 0.10\sin\left(\frac{5\pi}{10}t\right) + \underbrace{1.11\sin\left(\frac{7\pi}{10}t\right)}_{\text{large coefficient}} - 0.05\sin\left(\frac{9\pi}{10}t\right) + \cdots$$

$\frac{7\pi}{10} \approx 2.199$ fundamental freq $\sqrt{5} \approx 2.236$

"near" or "practical" resonance


Homogeneous
Steady Periodic
General