9.4 Application of Fourier Series (part 2)

$mx'' + cx' + kx = F(t)$ $F(t)$ is periodic

last time: $c = 0$, there is a chance of resonance
$F(t)$ has same frequency as $\sqrt{\frac{k}{m}}$ (fund. freq.)

for example,

$$x'' + 9x = \begin{cases} 1 & 0 < t < \pi \\ -1 & \pi < t < 2\pi \end{cases}$$

$$= \dots \text{ as sine series } = \sum_{n=1}^{\infty} \frac{2}{n}[1-(-1)^n] \sin(nt)$$

expand x as the same type of series (sine, $L = \pi$)
but unknown coefficients

$x = \sum_{n=1}^{\infty} B_n \sin(nt)$ sub into the eq. $x'' + 9x = \sum_{n=1}^{\infty} \frac{2}{n}[1-(-1)^n] \sin(nt)$

$\vdots$

$$B_n = \frac{2[1-(-1)^n]}{n(9-n^2)} \sin(nt)$$

general solution

$$x = C_1\cos(3t) + C_2\sin(3t) + \underbrace{\sum_{n=1}^{\infty} \frac{2\left[1-(-1)^n\right]}{n(9-n^2)}\sin(nt)}_{\text{steady periodic solution}}$$

$B_3 \to \infty$ $(n=3)$ $\quad$ $\sin(3t)$ is duplicating the fundamental solution

proper way to handle:

Solve $x'' + 9x = At\cos 3t + Bt\sin 3t$ (undetermined coefficients)

$$\text{plus} \quad x_p = \sum_{\substack{n=1 \\ n\neq 3}}^{\infty} \frac{2\left[1-(-1)^n\right]}{n(9-n^2)}\sin(nt)$$

after doing all that, we get

$$x = C_1\cos 3t + C_2\sin 3t - \frac{2}{3}t\cos 3t + \sum_{\substack{n=1 \\ n\neq 3}}^{\infty} \frac{2\left[1-(-1)^n\right]}{n(9-n^2)}\sin(nt)$$

<u>resonance</u> $\quad x \to \infty$ as $t \to \infty$ because

to mitigate that, let's add a damper $(c>0)$

$$mx'' + cx' + kx = F(t)$$

expand the periodic $F(t)$ as a sine series again

$$F(t) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi t}{L}\right)$$

because of the presence of x', we need to expand x as a full Fourier series

$$x = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi t}{L}\right) + B_n \sin\left(\frac{n\pi t}{L}\right)$$

$$x' = \sum_{n=1}^{\infty} -\frac{n\pi}{L} A_n \sin\left(\frac{n\pi t}{L}\right) + \frac{n\pi}{L} B_n \cos\left(\frac{n\pi t}{L}\right)$$

$$x'' = \cdots$$

sub all into $mx'' + cx' + kx = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi t}{L}\right)$

equate coefficients of $\cos\left(\frac{n\pi t}{L}\right)$ and $\sin\left(\frac{n\pi t}{L}\right)$

we get $\qquad \omega_n = \frac{n\pi}{L}$

$$A_n = \frac{-F_n c\,\omega_n}{(k-m\omega_n^2)^2 + (c\omega_n)^2} \qquad (A_0 = 0)$$

$$B_n = \frac{F_n(k-m\omega_n^2)}{(k-m\omega_n^2)^2 + (c\omega_n)^2}$$

note as long as $c \neq 0$,
the denominators $\neq 0$
none of the A_n or $B_n \to \infty$
for any n

Steady periodic solution

$$x_p = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi t}{L}\right) + B_n \sin\left(\frac{n\pi t}{L}\right)$$

rewrite in different form to match the sine $F(t)$ series

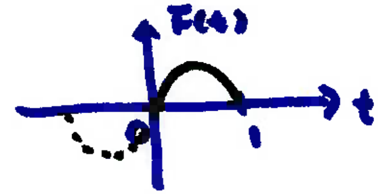
$$A_n \cos(\omega_n t) + B_n \sin(\omega_n t) = C_n \sin(\omega_n t - \alpha_n)$$

$$C_n = \frac{F_n}{\sqrt{(k-m\omega_n^2)^2 + (c\omega_n)^2}} \qquad \alpha_n = \tan^{-1}\left(\frac{c\omega_n}{k-m\omega_n^2}\right)$$

$$x_p = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi t}{L} - \alpha_n\right)$$ compare to $$F(t) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi t}{L}\right)$$

damper keeps the frequencies in $F(t)$ the same but
each has a phase shift (delay)

<u>example</u> $x'' + x' + 13x = F(t)$ $F(t) = t - t^2$ $0 < t < 1$ period 2



add odd extension
$\rightarrow$ sine series

$$F(t) = \cdots = \sum_{n=1}^{\infty} \frac{4}{n^3\pi^3}\left[1 - (-1)^n\right] \sin(n\pi t)$$

using the values of $m, k,$ and c, and the formulas from last page, we get this <u>steady periodic solution</u>
(complementary $\to 0$ as $t \to \infty$)

$$x_p = \sum_{n=1}^{\infty} \frac{\frac{4}{n^3\pi^3}\left[1-(-1)^n\right]}{\sqrt{(13-(n\pi)^2)^2 + (n\pi)^2}} \sin\left(n\pi t - \tan^{-1}\left(\frac{n\pi}{13 - n^2\pi^2}\right)\right)$$

$$= 0.0582 \sin(\pi t - 0.7892) + 0.000125 \sin(3\pi t - 3.0179)$$

$$+ 0.000009 \sin(5\pi t - 3.0745) + \cdots$$

dominant frequency is π (largest coefficient when $n = 1$)