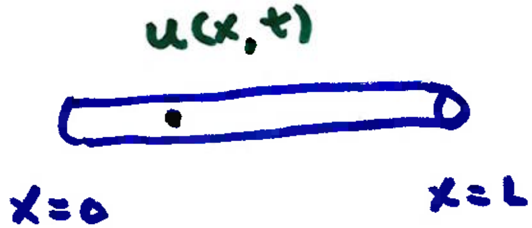
9.5 Heat Equation and Separation of Variables

conducting (metal) rod of length L

ends kept at certain temperatures

heat up the metal rod, then remove the heat source



the rod is laterally insulated

(heat only flows along the rod)

temp at x, t is $u(x,t)$

function of position and time

the equation that gives us $u(x,t)$ is the 1-D Heat Equation

only along the rod
(x-direction)

$$\boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}}$$

k: constant based on conductivity of material

or $\boxed{u_t = k u_{xx}}$

this is a partial differential eq (PDE)

$\frac{\partial}{\partial t}, \frac{\partial}{\partial x}$ as opposed $y'' + 2y' + y = 0$ (ODE)

1-D Heat Eq.

$u_t = k u_{xx}$

$u(0,t) = T_1$ temp at left end $(x=0)$
$u(L,t) = T_2$ temp at right end $(x=L)$ } Boundary conditions

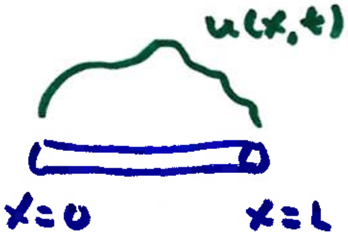
↓

if $T_1 = T_2 = 0$ the BC's are homogeneous

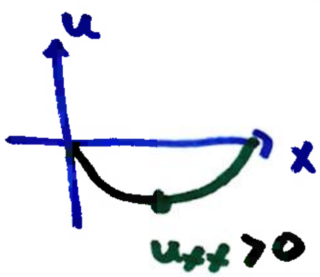
$u(x,0) = f(x)$ initial temp. profile $(t=0)$

goal: find $u(x,t)$

physical meaning of $u_t = k u_{xx}$



if $u_{xx} > 0$ temp. profile is concave up



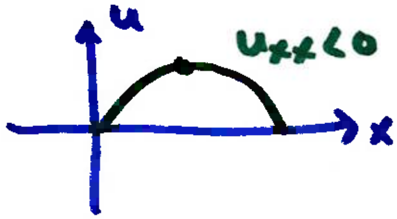
temp. is ~~higher~~ lower than avg. temp. of the surrounding

→ heat wants to come to this point

temp. here increase over time $u_t > 0$

$u_{xx} < 0$, temp. is higher than nearby, so temp. decreases over time

$u_t < 0$



$u_{xx} = 0 \to u_t = 0$ steady-state

$u_t = K u_{xx}$ $\quad u(x,t)$

solution method: separation of variables

assume $u(x,t)$ is a product of two functions each depending on only x or t

$$u(x,t) = X(x)\,T(t)$$

$$\frac{\partial u}{\partial t} = u_t = X(x)\,T'(t) \qquad \frac{\partial^2 u}{\partial x^2} = u_{xx} = X''(x)\,T(t)$$

put into $u_t = K u_{xx}$

$$X T' = K X'' T$$

$$\frac{X''}{X} = \frac{\cancel{K}\,T'}{KT}$$ only possible if the ratio is ALWAYS a constant

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda \quad (\lambda > 0) \quad \text{separation constant}$$

this results in <u>two</u> ODE's

$$\frac{X''}{X} = -\lambda \rightarrow \boxed{X'' + \lambda X = 0}$$

$$\frac{T'}{kT} = -\lambda \rightarrow \boxed{T' + \lambda k T = 0}$$

} need $X(0)$, $X(L)$ to solve completely

BC's of PDE: $u(0,t) = T_1 = 0$ (homogeneous)

$$X(0)T(t) = 0 \rightarrow \boxed{X(0) = 0}$$

$u(L,t) = T_2 = 0$ (homogeneous)

$$X(L)T(t) = 0 \rightarrow \boxed{X(L) = 0}$$

now solve $X'' + \lambda X = 0$, $X(0) = X(L) = 0$
$(\lambda > 0)$

$X(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$

$X(0) = 0 \rightarrow \quad 0 = c_1$

$X(L) = 0 \rightarrow \quad 0 = c_2 \sin(\sqrt{\lambda}\, L)$

require $c_2 \neq 0$ so we have a nontrivial solution

$\sin(\sqrt{\lambda}\, L) = 0$

$\sqrt{\lambda}\, L = n\pi \qquad n = 1, 2, 3, \ldots$

so, $\lambda = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, 3$

$$\boxed{\lambda_n = \frac{n^2 \pi^2}{L^2}}$$ eigenvalues

for each n, X has one solution

$X_n(x) = \sin(\sqrt{\lambda_n}\, x)$

$$\boxed{X_n = \sin\left(\frac{n\pi}{L} x\right)}$$ eigenfunctions

now $T' + k\lambda T = 0$

$$T' + \frac{kn^2\pi^2}{L^2} T = 0 \qquad \text{1st-order ODE}$$

$$T(t) = c_1 e^{-\frac{kn^2\pi^2}{L^2} t}$$

for each n, there is one solution $T_n(t)$

$$\boxed{T_n(t) = e^{-\frac{kn^2\pi^2}{L^2} t}} \qquad \text{eigenfunctions}$$

$$u(x,t) = XT$$

for each n, one solution $u_n(x,t) = X_n(x)\, T_n(t)$

$$= e^{-\frac{kn^2\pi^2}{L^2} t} \sin\left(\frac{n\pi}{L} x\right)$$

superposition: general solution is ~~sum~~ linear combo of all of u_n

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{kn^2\pi^2}{L^2} t} \sin\left(\frac{n\pi}{L} x\right)} \qquad b_n = ?$$

initial condition: $u(x,0) = f(x)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$ sine series

$$\boxed{b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx}$$

<u>example</u>

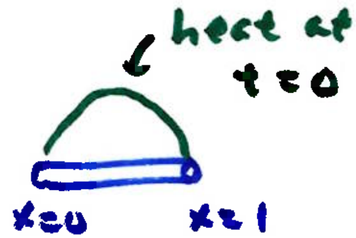
$u_t = 2u_{xx}$

$k = 2$

$0 < x < 1 \quad L = 1$

$u(0,t) = u(L,t) = 0$

$u(x,0) = 9\sin(\pi x) - \frac{1}{4}\sin(3\pi x)$

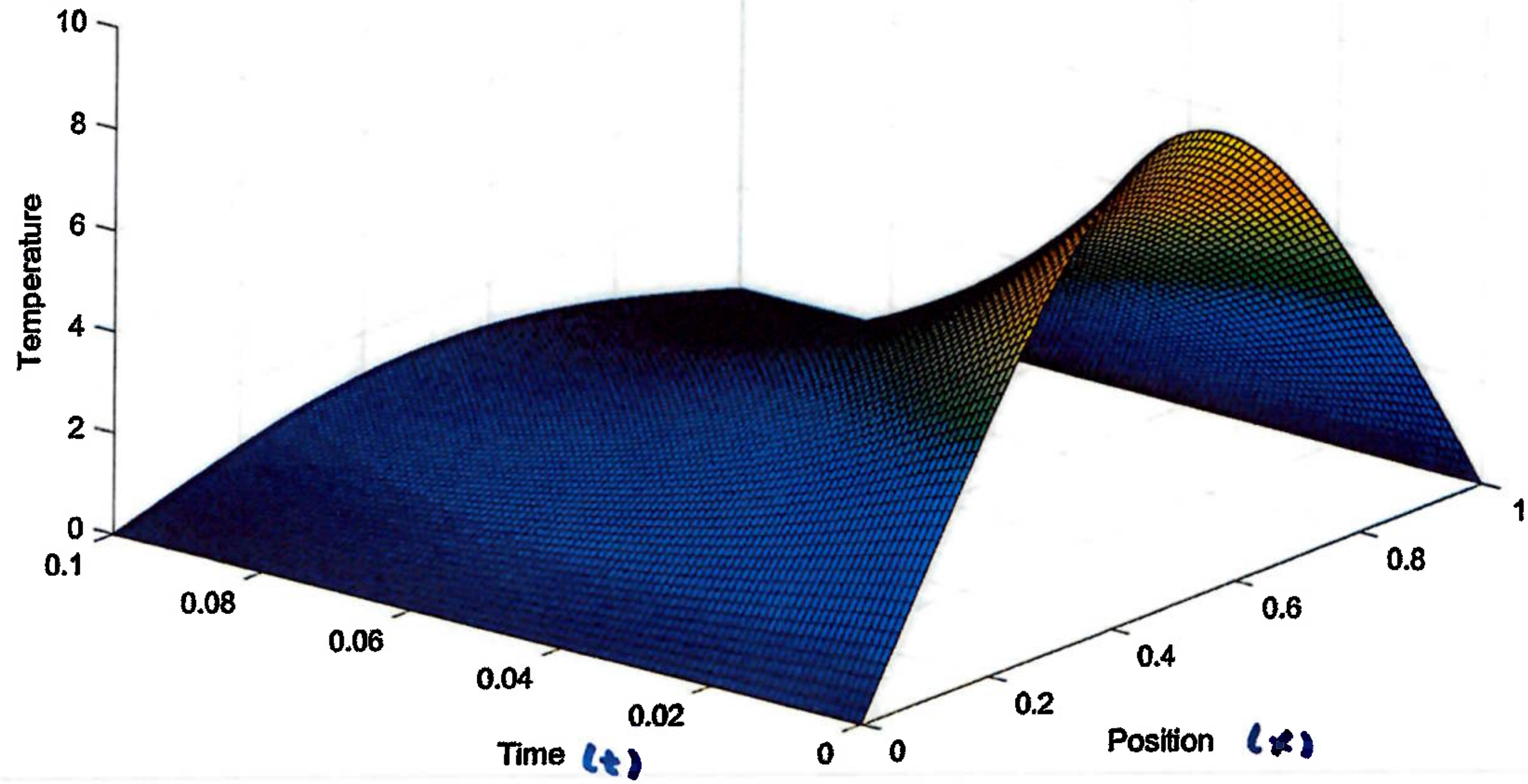


⋮

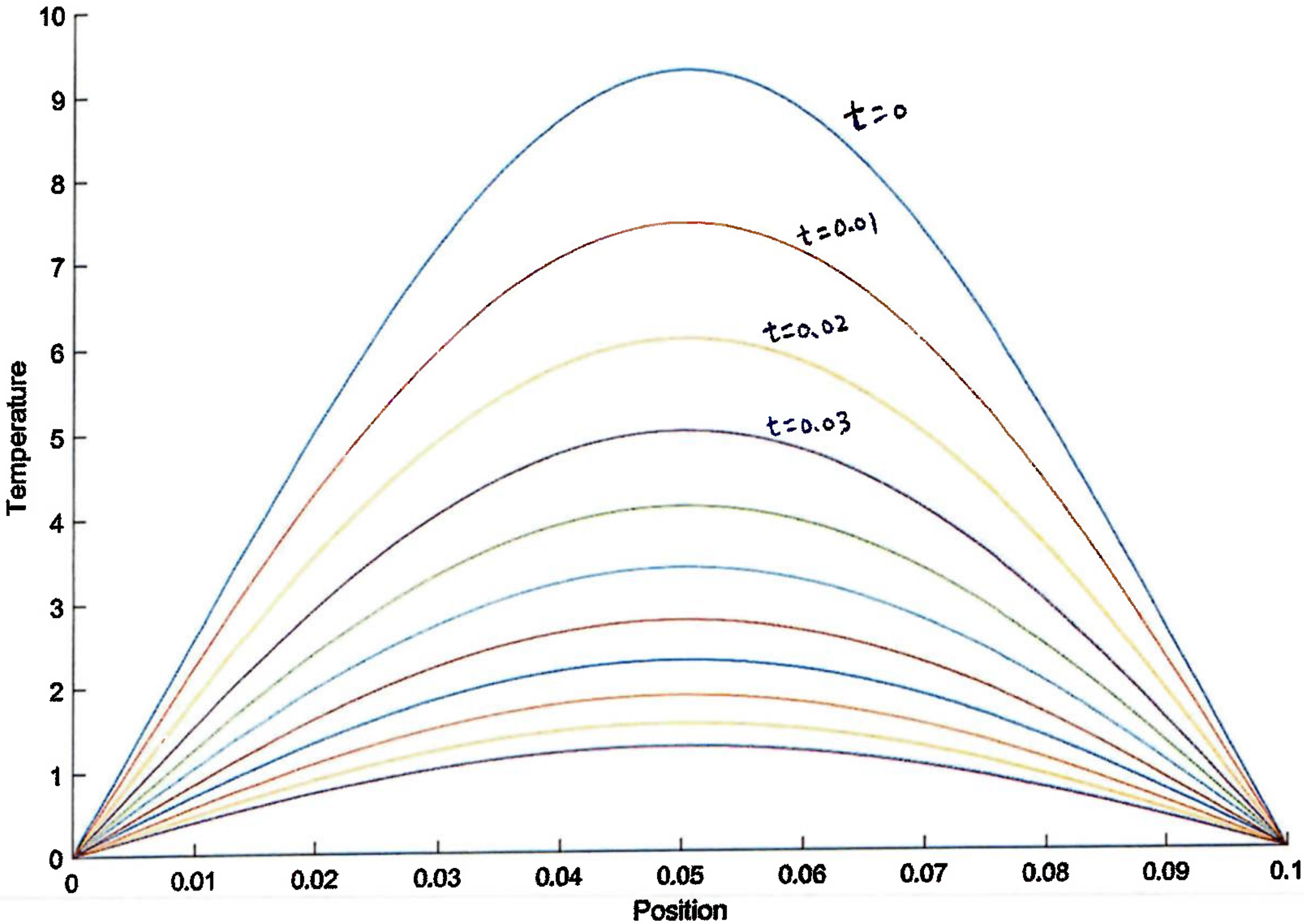
$b_1 = 9 \qquad b_3 = -\frac{1}{4} \qquad$ all others 0

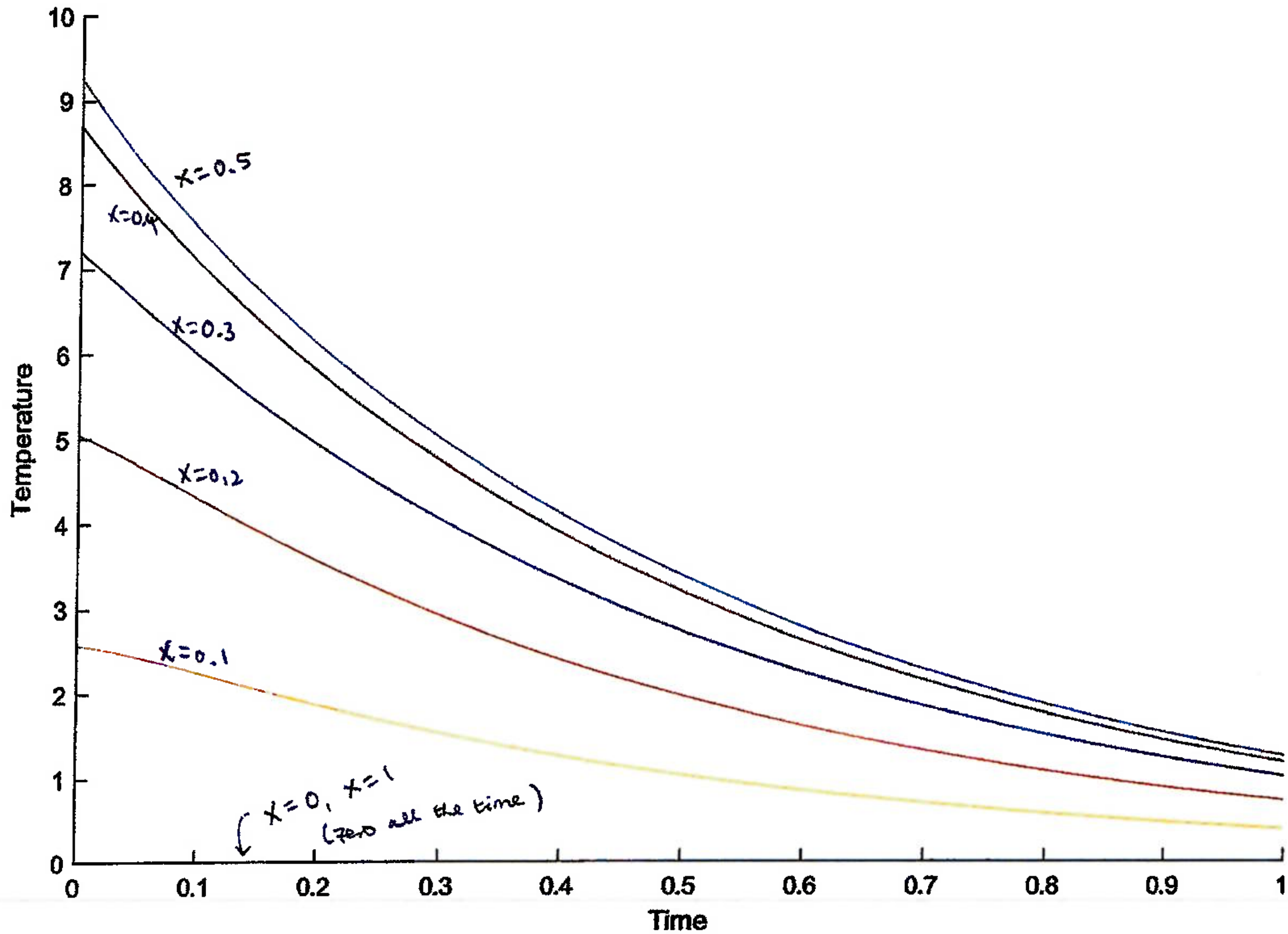
$$u(x,t) = 9e^{-2\pi^2 t}\sin(\pi x) - \frac{1}{4}e^{-18\pi^2 t}\sin(3\pi x)$$

graph of 2-variable function is a surface



Temperature
10
8
6
4
2
0
0.1
0.08
0.06
0.04
0.02
0
Time (t)
0
0.2
0.4
0.6
0.8
1
Position (x)


t=0
t=0.01
t=0.02
t=0.03
Temperature
Position
10
9
8
7
6
5
4
3
2
1
0
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1



10
9
8
7
6
5
4
3
2
1
0
Temperature
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Time
x=0.5
x=0.4
x=0.3
x=0.2
x=0.1
x=0, x=1
(zero all the time)