9.5 Heat Equation (part 2)

~~last~~: last time

$u_t = k u_{xx} \qquad 0 < x < L$

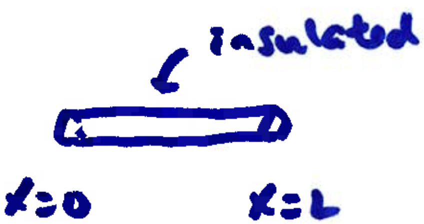
$u(0,t) = 0$
$u(L,t) = 0$ } homogeneous boundary conditions

$u(x,0) = f(x)$ initial heating profile

Solution:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 k \pi^2 t / L^2} \sin\left(\frac{n\pi x}{L}\right)$$

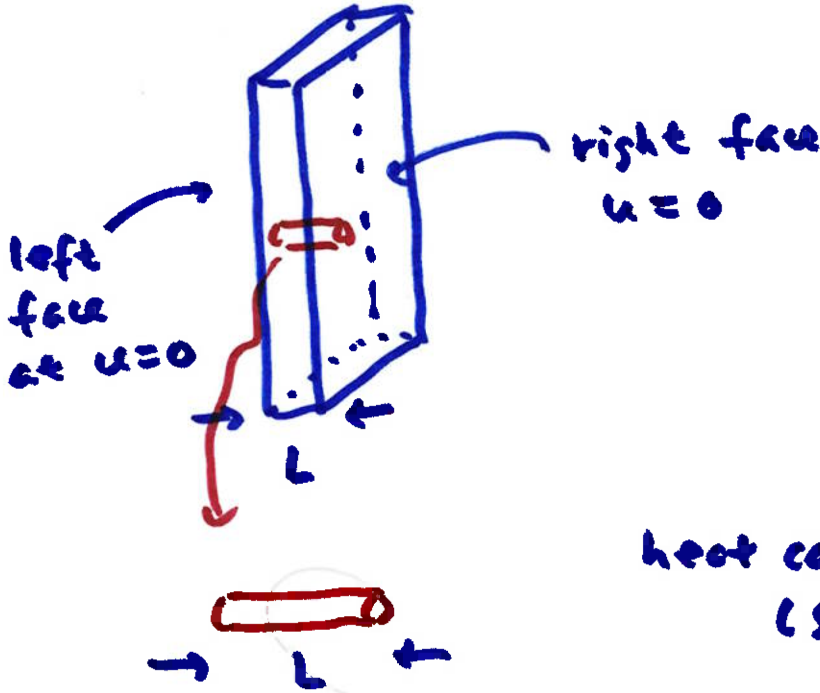
$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$



not just applicable to a rod

for example, a slab made of one material

L: width of slab



heat cannot travel "up" or "down"
(same material, same heat property)

same core sample anywhere in the slab

this core sample is ~~so~~ described by
the heat eq. we solved

→ we can use that to understand
how heat move around in the slab

example copper slab ($k = 1.15\ cm^2/s$)

both faces at $0°C$

thickness is 4 cm ($L = 4$ cm)

initially, the entire interior is heated to $100°C$

Find: temp. at center 3 seconds later

how long before the center is cooled to $5°C$?

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

BC's: $u(0,t) = 0$

$u(L,t) = 0$

IC: $u(x,0) = 100 = f(x) \rightarrow b_n = \cdots = \frac{200}{n\pi}\left[1-(-1)^n\right]$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{200}{n\pi}\left[1-(-1)^n\right] e^{-1.15 n^2\pi^2 t/(4)^2} \sin\left(\frac{n\pi x}{4}\right)$$

temp at center of slab $(x=2)$ 3 seconds later $(t=3)$

$u(2,3)=$ infinite series, need to truncated and approx.

negative exponential → fast converging series

only need a small number of terms

here, one-term approx. is

$u(2,3) \approx 15.16\,°C$

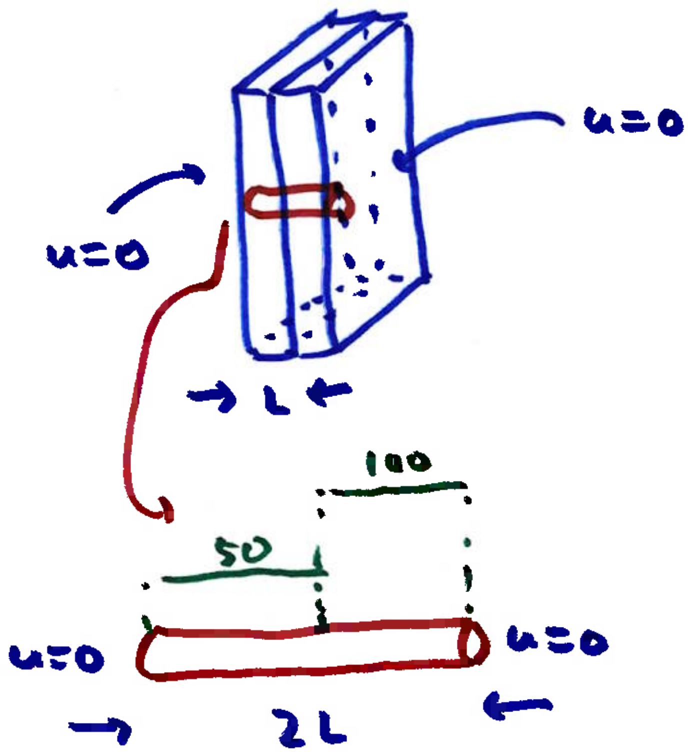
time to cool to $5°C$ → even longer t, so negative exp. means one-term is very good.

$$u(x,t)=\frac{400}{\pi}e^{-1.15\pi^2 t/16}\sin\left(\frac{\pi x}{4}\right)$$

want $u(2,t)=5$

solve for t: $t \approx 4.6$ seconds

another variation: two slabs



each is of uniform material

both made of same material

we can heat each differently at $t=0$,

for example, 50°C left

100°C right

length 2L

IC: $f(x) = \begin{cases} 50 & 0 < x < L \\ 100 & L < x < 2L \end{cases}$

re-use the same solution except $L \to 2L$

now, instead of specifying left and right at 0 temp, let's use the boundary condition that says the ends are also insulated.

$u_t = k u_{xx} \qquad 0 < x < L$

$x=0$ $\qquad$ $x=L$

BC's: $\underline{u_x(0,t) = u_x(L,t) = 0}$

no heat flow in x-direction at ends

IC: $u(x,0) = f(x)$ (same as before)

separation of variables: $u(x,t) = X(x)T(t)$

$u_t = k u_{xx}$ turns into

$$X T' = k X'' T$$

separate the same way

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda \quad \text{(constant)}$$

same as before

$$X'' + \lambda X = 0$$

BC's: $u_x(0,t) = 0 \to X'(0)T(t) = 0 \to X'(0) = 0$

$u_x(L,t) = 0 \to X'(L)T(t) = 0 \to X'(L) = 0$

$$X = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

$$X' = -\sqrt{\lambda}\, c_1 \sin(\sqrt{\lambda}x) + \sqrt{\lambda}\, c_2 \cos(\sqrt{\lambda}x)$$

$X'(0) = 0 \to \quad 0 = c_2\sqrt{\lambda} \quad \to \quad c_2 = 0 \quad (\text{or } \lambda = 0)$

$X'(L) = 0 \to \quad 0 = -\sqrt{\lambda}\, c_1 \sin(\sqrt{\lambda}L) \qquad c_1 \neq 0$ (else trivial)

$$\sin(\sqrt{\lambda}L) = 0$$

$$\to \sqrt{\lambda}L = n\pi$$

$$\boxed{\lambda_n = \frac{n^2\pi^2}{L^2}}$$

same eigenvalues

sin for non-insulated

$$\boxed{X_n = \cos\left(\frac{n\pi x}{L}\right)}$$

$n = \underline{\underline{0}}, 1, 2, 3, \ldots$

non-insulated $n \neq 0$

same T solution : $\boxed{T_n = e^{-kn^2\pi^2 t/L^2}}$

for each n, one solution: $u_n(x,t) = e^{-kn^2\pi^2 t/L^2} \cos\left(\frac{n\pi x}{L}\right)$

$$u_0 = 1$$

general solution is linear combination of all of those

$$\boxed{u(x,t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n e^{-kn^2\pi^2 t/L^2} \cos\left(\frac{n\pi x}{L}\right)}$$

insulated ends

IC: $u(x,0) = f(x)$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

cosine series

$$\boxed{a_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx}$$