# 9.5 Heat Equation (part 3)

**non-insulated**


$$u_t = k u_{xx} \qquad 0 < x < L$$

$$\underline{u(0,t) = u(L,t) = 0}$$ Homogeneous BC's

$$u(x,0) = f(x)$$

→ fix end temps

~~Diri~~ Dirichlet condition

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

**insulated**


$$u_t = k u_{xx} \qquad 0 < x < L$$

$$\underline{u_x(0,t) = u_x(L,t) = 0}$$

fix the derivatives

Neumann Condition

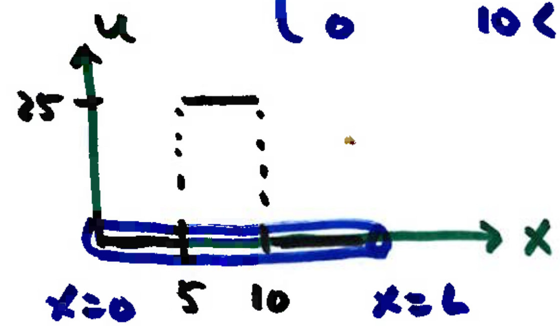
$$u(x,0) = f(x)$$

$$u(x,t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n e^{-kn^2\pi^2 t/L^2} \cos\left(\frac{n\pi x}{L}\right)$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

example Insulated ends. $L = 30$, $k = 1$

initial temp. profile $f(x) = \begin{cases} 0 & 0 < x < 5 \\ 25 & 5 < x < 10 \\ 0 & 10 < x < 30 \end{cases}$




we don't control the temps at ends
we control the heat flow through ends (zero)

solution:

$$u(x,t) = \frac{25}{6} + \sum_{n=1}^{\infty} \frac{50}{n\pi}\left[\sin\left(\frac{n\pi}{3}\right) - \sin\left(\frac{n\pi}{6}\right)\right] e^{-n^2\pi^2 t/900} \cos\left(\frac{n\pi x}{30}\right)$$

$$= \frac{25}{6} + 3.5 e^{-\pi^2 t/900}\cos\left(\frac{\pi x}{30}\right) - 3.2 e^{-9\pi^2 t/900}\cos\left(\frac{3\pi x}{30}\right)$$

$$- 4.1 e^{-16\pi^2 t/900}\cos\left(\frac{4\pi x}{30}\right) + \cdots$$

$\dfrac{-n^2\pi^2 t}{900}$ is not small if $t$ is not small ($k$ small or $L$ large)

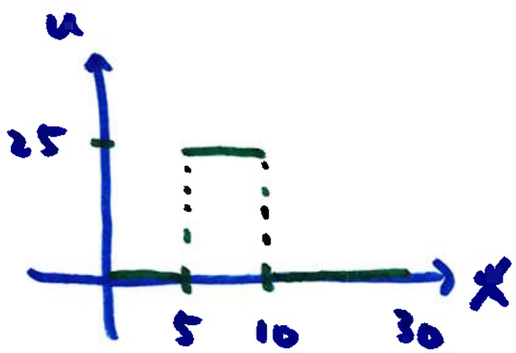
→ heat doesn't travel through fast

→ series converges slowly (many terms needed unless $t$ is very large)

note if $t \to \infty$, $u \to \frac{25}{6}$ (same temp all along the rod)

Steady-state solution

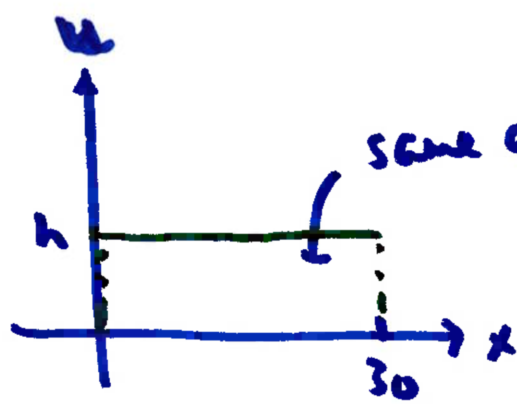
why $\frac{25}{6}$?




"total amount" of heat

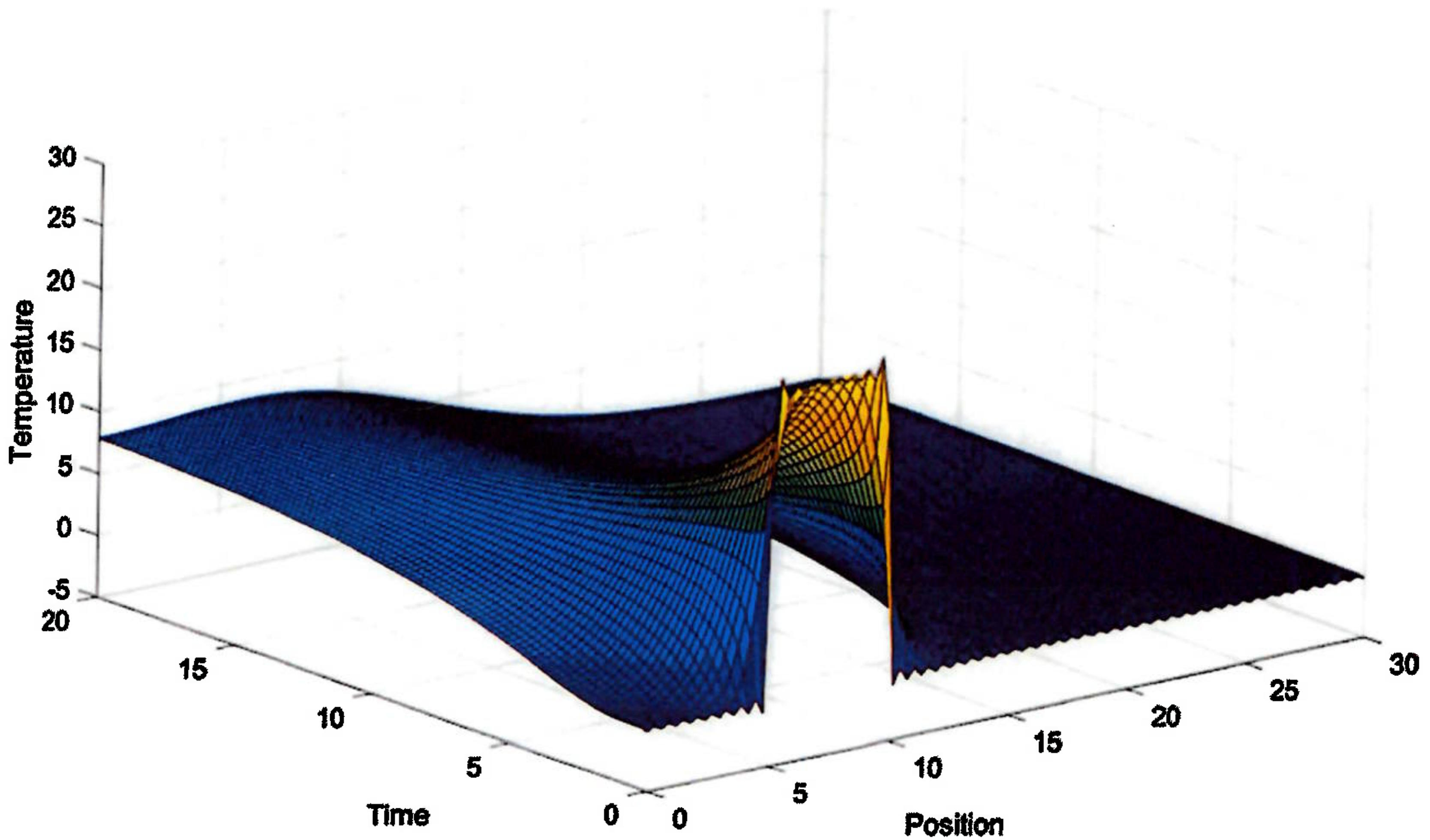
$= (25)(5) = 125$

heat cannot escape but doesn't want to be uneven




same amount of heat $= 125$

$h(30) = 125 \to h = \frac{25}{6}$



Temperature
30
25
20
15
10
5
0
-5
20
15
10
5
0
Time
0
5
10
15
20
25
30
Position

now revisit the non-insulated problem, but allow the ~~two~~ end temps. to be non-zero


$$u_t = k u_{xx} \qquad 0 < x < L$$

$$\left.\begin{aligned} u(0,t) &= T_1 \\ u(L,t) &= T_2 \end{aligned}\right\} \text{nonhomogeneous BC's}$$

$$u(x,0) = f(x)$$

we can find the <u>steady-state</u> solution very quickly

↳ time is no longer a factor

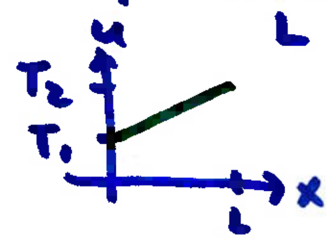
$$u_t = 0$$

$$u_t = k u_{xx} \rightarrow u_{xx} = 0 \rightarrow u = c_1 x + c_2$$

BC's: $u(0,t) = T_1 \rightarrow T_1 = c_1 \cdot 0 + c_2 \rightarrow c_2 = T_1$

$u(L,t) = T_2 \rightarrow T_2 = c_1 L + T_1 \rightarrow c_1 = \dfrac{T_2 - T_1}{L}$

steady-state: $\boxed{u = \dfrac{T_2 - T_1}{L} x + T_1}$



that's just the steady-state. We want $u(x,t)$ for all $t$

$u_t = K u_{xx}$

let $w(x,t) = \underbrace{u(x,t) - \left[ \frac{T_2 - T_1}{L} x + T_1 \right]}_{\text{subtract steady-state } V(x)} = u(x,t) - V(x)$

$w_t = u_t$ $\quad$ $w_{xx} = u_{xx}$ $\quad$ so, $w(x,t)$ also satisfies the heat eq.

$w_t = K w_{xx}$

BC's: $u(0,t) = T_1 \rightarrow w(0,t) = u(0,t) - T_1 = T_1 - T_1 = 0$

$u(L,t) = T_2 \rightarrow w(L,t) = u(L,t) - T_2 = T_2 - T_2 = 0$

$w$ has homogeneous BC's

(can reuse the homogeneous solution)

IC: $u(x,0) = f(x)$

$\rightarrow w(x,0) = u(x,0) - V(x)$

$= f(x) - V(x)$

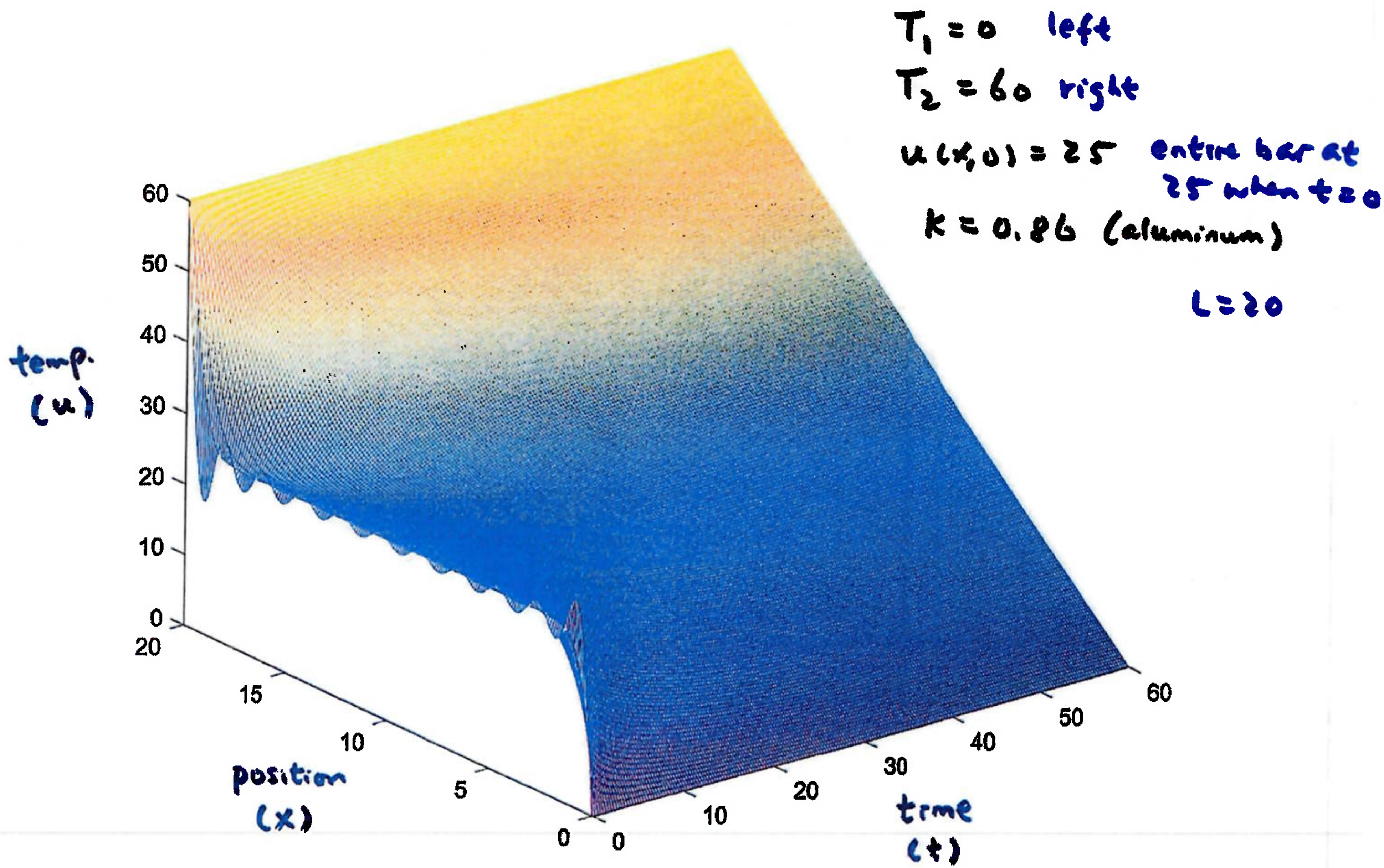
reuse the homogeneous solution w/ different IC

$$w(x,t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

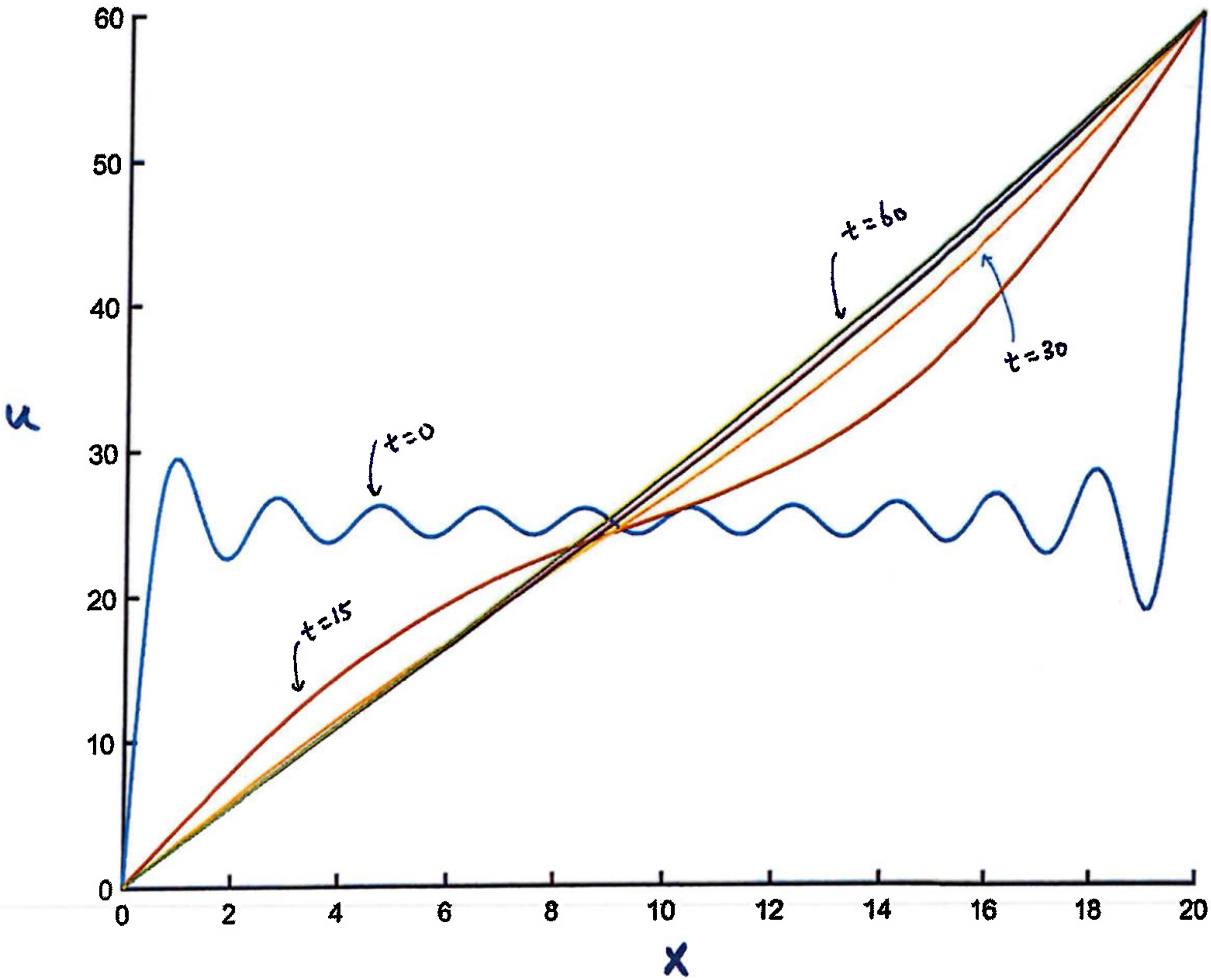
$$b_n = \frac{2}{L}\int_0^L \left[f(x) - \left(\frac{T_2 - T_1}{L}x + T_1\right)\right] \sin\left(\frac{n\pi x}{L}\right) dx$$

$w = u - v$

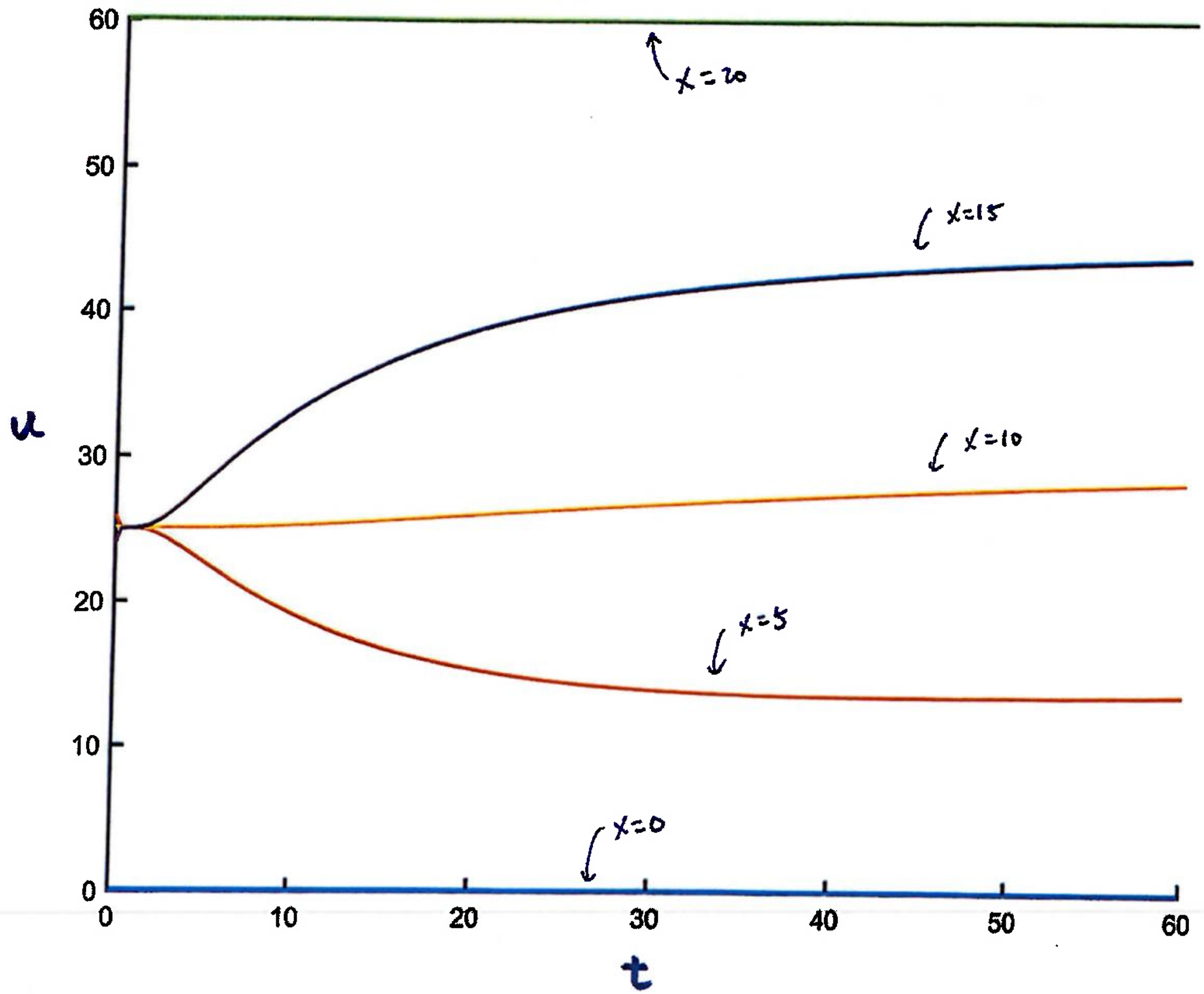
$$u(x,t) = \left(\frac{T_2 - T_1}{L}x + T_1\right) + \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$



$T_1 = 0$ left
$T_2 = 60$ right
$u(x,0) = 25$ entire bar at 25 when $t=0$
$k = 0.86$ (aluminum)
$L = 20$
temp. (u)
60
50
40
30
20
10
0
position (x)
20
15
10
5
0
time (t)
0
10
20
30
40
50
60



60
50
40
30
20
10
0
0
2
4
6
8
10
12
14
16
18
20
u
x
t=0
t=15
t=30
t=60



60
50
40
30
20
10
0
u
x=20
x=15
x=10
x=5
x=0
0
10
20
30
40
50
60
t