9.6 Wave Equation

String length L fixed at both ends

not disturbed - equilibrium

y: displacement from equilibrium

$y>0 \rightarrow$ above eq.

$y<0 \rightarrow$ below

the governing eq: $\dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2} \longleftrightarrow y_{tt} = a^2 y_{xx}$

$a^2 \rightarrow$ how fast wave propagates

$a^2 = \dfrac{\text{tension}}{\text{density}}$

What does $y_{tt} = a^2 y_{xx}$ physically represent?

$y_{xx} < 0$ (concave down)

$y_{tt} < 0 \rightarrow$ acceleration / force downward

$y_{xx} > 0$ (concave up)

$y_{tt} > 0 \rightarrow$ upward force

$y_{xx} = 0 \rightarrow$ no acceleration

string wants to go back
to equilibrium
(just like a spring)

$y_{tt} = a^2 y_{xx}$ is like the heat eq. it needs boundary and initial conditions

BC's: $y(0,t) = y(L,t) = 0$ ends fixed at 0 Dirichlet condition

or

wrong discussion in lecture →

$\cancel{y}(0,t) = \cancel{y}(L,t) = 0$ no ~~velocity~~ slope at ends Neumann condition

y_x y_x

IC's: (two needed because 2nd-order deriv. of t)

$y(x,0) = f(x)$ initial displacement

$y_t(x,0) = g(x)$ " velocity

today: Problem A (no initial velocity, non zero displacement)

$y_{tt} = a^2 y_{xx}$ $\quad 0 < x < L$

$y(0,t) = y(L,t) = 0$ fixed ends

$y_t(x,0) = 0$ initial velocity

$y(x,0) = f(x)$ displacement

Solve it like the heat eq: separation of variables

$$y(x,t) = X(x)\,T(t)$$

$$y_{tt} = XT'' \qquad y_{xx} = X''T$$

$$XT'' = a^2 X'' T$$

$$\frac{X''}{X} = \frac{T''}{a^2 T} = \text{constant} = -\lambda$$

separation constant just like in heat eq.

two ODEs come out

$$X'' + \lambda X = 0$$

BC's: $y(0,t)=0 \to X(0)T(t)=0 \to X(0)=0$

$y(L,t)=0 \to X(L)T(t)=0 \to X(L)=0$

exactly like heat eq.

$$X(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

$X(0)=0 \to c_1 = 0$

$X(L)=0 \to c_2 \sin(\sqrt{\lambda}L) = 0$ $\quad c_2 \neq 0$

$\sin(\sqrt{\lambda}L) = 0$

$\sqrt{\lambda}L = n\pi \qquad n = 1, 2, 3, \dots$

$\lambda = \frac{n^2\pi^2}{L^2}$ eigenvalues

$$\boxed{\lambda_n = \frac{n^2\pi^2}{L^2}}$$

solutions:

$$\boxed{X_n = \sin\left(\frac{n\pi x}{L}\right)}$$

eigenfunctions

now T equation: $\frac{T''}{a^2 T} = -\lambda \to T'' + \lambda a^2 T = 0$

$$T'' + \frac{a^2 n^2 \pi^2}{L^2} T = 0$$

IC: $y_t(x,0) = 0$

$X(x)\,T'(0) = 0 \rightarrow T'(0) = 0$

this part is different
from the heat eq.
(this is 2nd-order)

$$T = c_1 \cos\left(\frac{an\pi}{L} t\right) + c_2 \sin\left(\frac{an\pi}{L} t\right)$$

only one condition to use: $T'(0) = 0$

$$T' = -\frac{an\pi}{L} c_1 \sin\left(\frac{an\pi}{L} t\right) + \frac{an\pi}{L} c_2 \cos\left(\frac{an\pi}{L} t\right)$$

$$T'(0) = 0 \rightarrow 0 = c_2 \frac{an\pi}{L} \rightarrow c_2 = 0$$

so, for each n, solution is $\cancel{T_n = c_n \ldots}$

$$\boxed{T_n = \cos\left(\frac{n\pi a t}{L}\right)}$$

eigenfunctions
for T

because $y(x,t) = X(x)\,T(t)$

for each n $(n = 1, 2, 3, \ldots)$ there is one solution

$$y_n = \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{X_n} \underbrace{\cos\left(\frac{n\pi a t}{L}\right)}_{T_n}$$

So, the general solution is a linear combination of all of them

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi a t}{L}\right) \sin\left(\frac{n\pi x}{L}\right)}$$

A_n comes from the last initial condition

$$y(x,0) = f(x)$$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{sine series}$$

$$\boxed{A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx}$$

each n is a <u>mode</u> or <u>harmonic</u> of the vibration