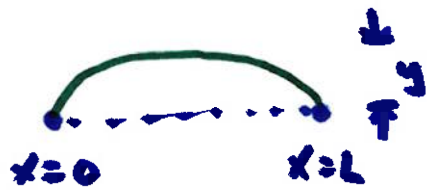
9.6 Wave Equation (part 2)

$$y_{tt} = a^2 y_{xx}$$



BC's: $y(0,t) = y(L,t) = 0$ Dirichlet Condition (end positions fixed)

(or) $y_x(0,t) = y_x(L,t) = 0$ Neumann Condition (slopes fixed)

IC's: $y(x,0) = f(x)$ initial displacement

$y_t(x,0) = g(x)$ initial velocity

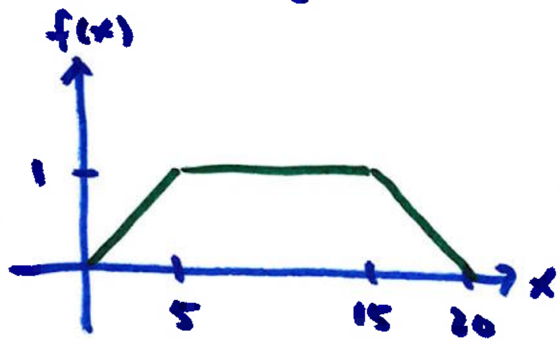
last time, we solved "Problem A" → $f(x) \neq 0$ but $g(x) = 0$

Solution: $$y(x,t) = \sum_{n=1}^{\infty} A_n \overbrace{\cos\left(\frac{n\pi a t}{L}\right)}^{\text{time}} \overbrace{\sin\left(\frac{n\pi x}{L}\right)}^{\text{space}}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

no negative exponential like heat → slow converging (if at all) series (many terms needed to approx well) example: $L=20$, $a=1$, $g(x)=0$

$$f(x)=\begin{cases}\frac{1}{5}x & 0\le x\le 5\\ 1 & 5\le x\le 15\\ \frac{20-x}{5} & 15\le x\le 20\end{cases}$$ initial displacement



$$A_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$=\frac{2}{20}\int_0^5 \frac{1}{5}x\sin\left(\frac{n\pi x}{20}\right)dx+\frac{2}{20}\int_5^{15} 1\cdot\sin\left(\frac{n\pi x}{20}\right)dx$$

$$+\frac{2}{20}\int_{15}^{20}\frac{20-x}{5}\sin\left(\frac{n\pi x}{20}\right)dx$$

$$=\cdots=\frac{8}{n^2\pi^2}\left(\sin\frac{n\pi}{4}+\sin\frac{3n\pi}{4}\right)\quad n=1,2,3,\ldots$$

$$y(x,t)=\sum_{n=1}^{\infty}\frac{8}{n^2\pi^2}\left(\sin\frac{n\pi}{4}+\sin\frac{3n\pi}{4}\right)\cos\left(\frac{n\pi t}{20}\right)\sin\left(\frac{n\pi x}{20}\right)$$

n=1 : Fundamental mode (first harmonic / overtone)

$$y_1(x,t) = \frac{8\sqrt{2}}{\pi^2}\cos\left(\boxed{\frac{\pi}{20}}t\right)\sin\left(\frac{\pi}{20}x\right)$$

↳ how fast string vibrates

fundamental frequency is $\frac{\pi}{20}$ rad/s

$$\frac{\pi/20}{2\pi} = \frac{1}{40}\text{ Hz}$$

if this were sound, we would "hear" a note at $\frac{1}{40}$ Hz

if this were sound, this is often the dominant one

n=2 : Second harmonic

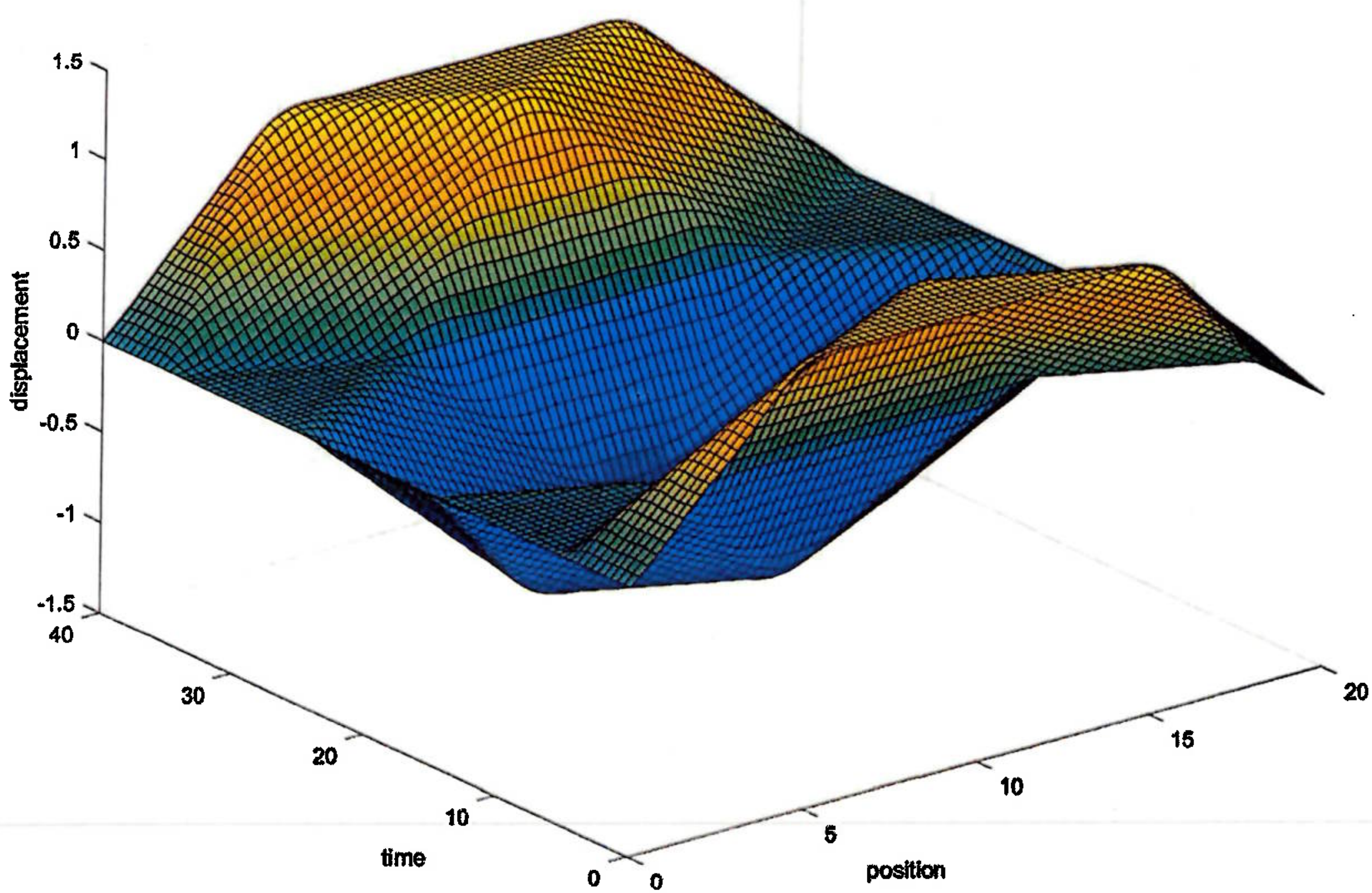
$$\text{freq} = 2\cdot\frac{\pi}{20} = \frac{\pi}{10}$$ doubling fundamental

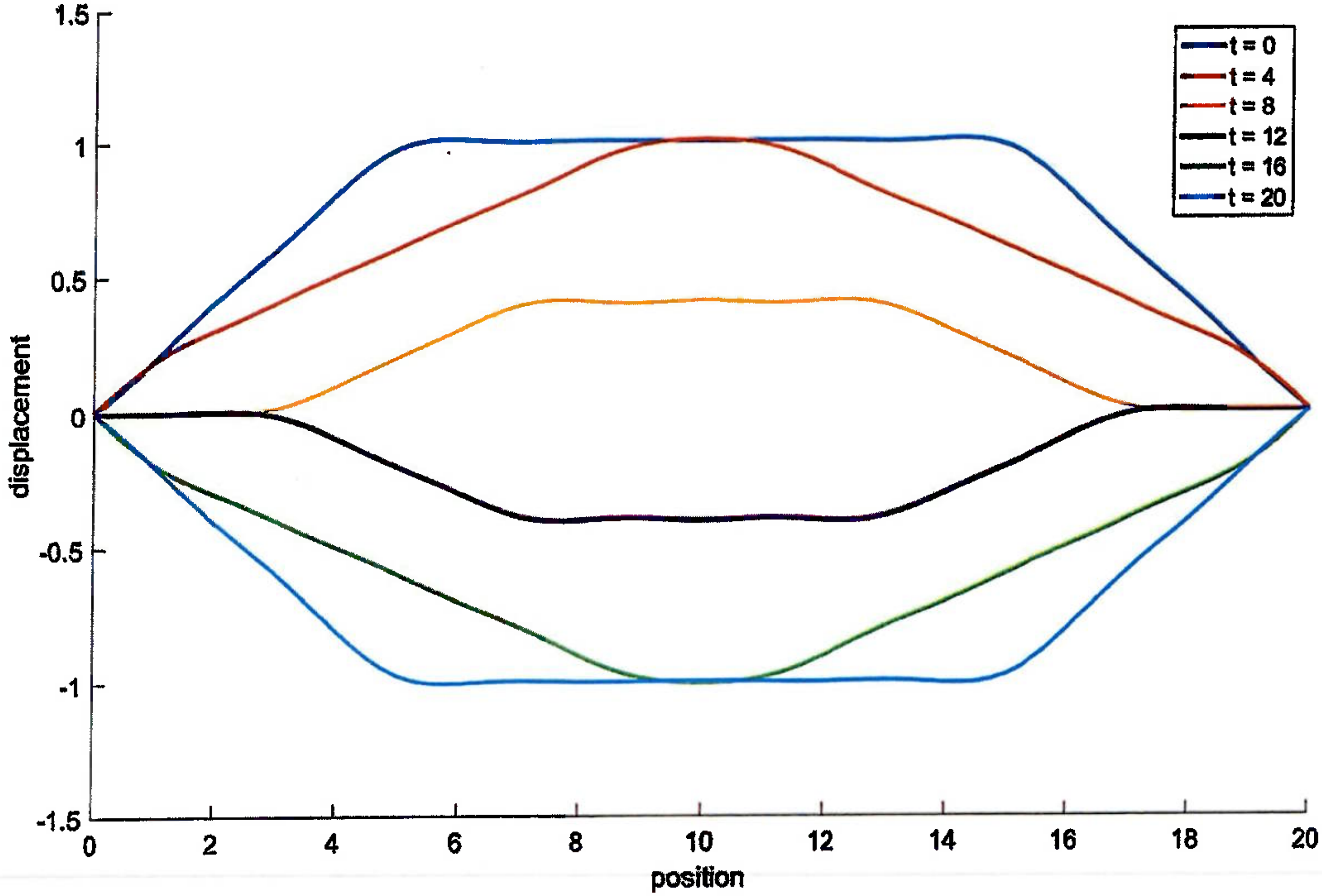
(one octave higher)

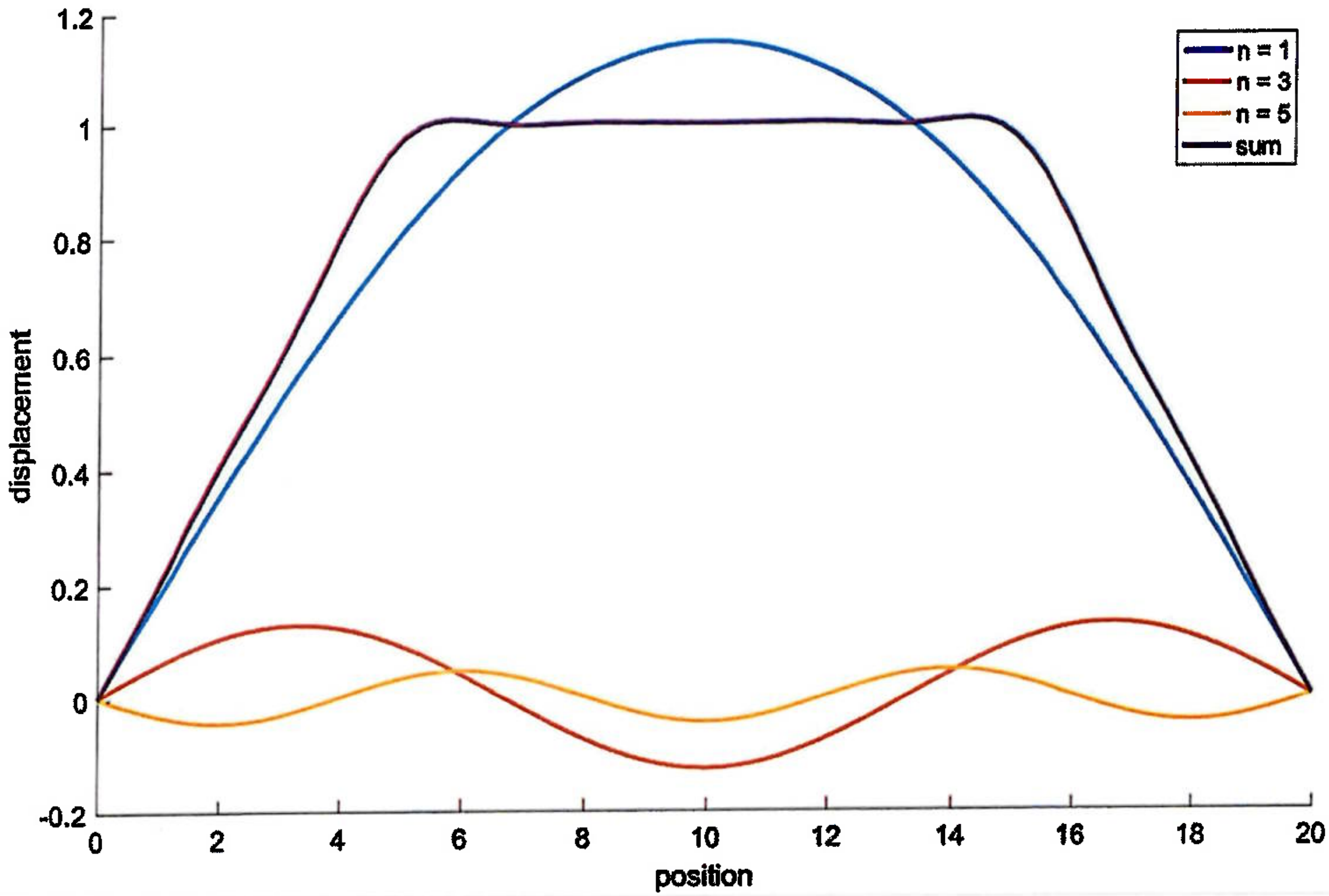
n=3 : third harmonic

3 times fundamental

(one octave and a perfect fifth above fundamental)


displacement
1.5
1
0.5
0
-0.5
-1
-1.5
time
40
30
20
10
0
position
0
5
10
15
20


t = 0
t = 4
t = 8
t = 12
t = 16
t = 20
1.5
1
0.5
0
-0.5
-1
-1.5
displacement
0
2
4
6
8
10
12
14
16
18
20
position



n = 1
n = 3
n = 5
sum
displacement
position
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
2
4
6
8
10
12
14
16
18
20

Problem B: no initial displacement
non zero initial velocity

$$y_{tt} = a^2 y_{xx}$$

BC's: $y(0,t) = y(L,t) = 0$

IC's: $y(x,0) = 0$ displacement

$y_t(x,0) = g(x)$ velocity

same process: separation of variables

$y = X(x)T(t)$ BC's: $y(0,t) = y(L,t) = 0$

$\hookrightarrow X(0) = X(L) = 0$

$$y_{tt} = a^2 y_{xx}$$

$$XT'' = a^2 X'' T$$

$$\frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda \rightarrow \begin{array}{l} X'' + \lambda X = 0 \quad X(0) = X(L) = 0 \\ T'' + a^2 \lambda = 0 \end{array}$$

same space solution (same BC's, will change if we used Neumann condition)

eigenvalues: $\lambda_n = \frac{n^2\pi^2}{L^2}$

eigenfunctions: $X_n = \sin\left(\frac{n\pi x}{L}\right)$

time part will change

$$T'' + a^2\lambda T = 0$$

$$T'' + \frac{a^2 n^2 \pi^2}{L^2} T = 0$$

$$T(t) = c_1 \cos\left(\frac{n\pi a t}{L}\right) + c_2 \sin\left(\frac{n\pi a t}{L}\right)$$

IC's : $y(x,0) = 0$ $\qquad$ $y_t(x,0) = g(x)$ $\qquad$ cos in Problem A

$$\downarrow$$

$$X(x)T(0) = 0 \rightarrow T(0) = 0$$

$0 = c_1$ $\qquad$ so for time, eigenfunctions are $T_n = \sin\left(\frac{n\pi a t}{L}\right)$

for each $n = 1, 2, 3, \ldots$ $\quad y_n = X_n T_n = \sin\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$

Superimpose all

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)}$$

the unused IC: $y_t(x,0) = g(x)$

$$y_t(x,t) = \sum_{n=1}^{\infty} \frac{n\pi a}{L} B_n \cos\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

$$g(x) = \underbrace{\sum_{n=1}^{\infty}\left[\frac{n\pi a}{L} B_n\right]\sin\left(\frac{n\pi x}{L}\right)}$$

sine series w/ coefficients $\frac{n\pi a}{L} B_n$

$$\frac{n\pi a}{L} B_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)$$

$$\boxed{B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$