9.6 Wave Equation (part 3)

last time: $y_{tt} = a^2 y_{xx}$ $\quad 0 < x < L$

$y(0,t) = y(L,t) = 0$ fixed ends

$y(x,0) = 0$ no initial displacement

$y_t(x,0) = g(x)$ initial velocity

Solution: $$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

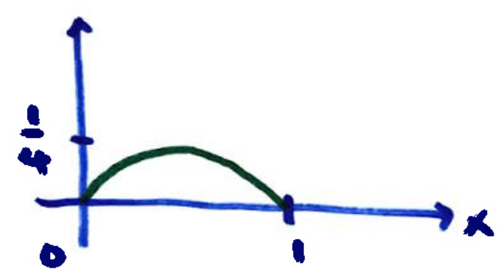
$$B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

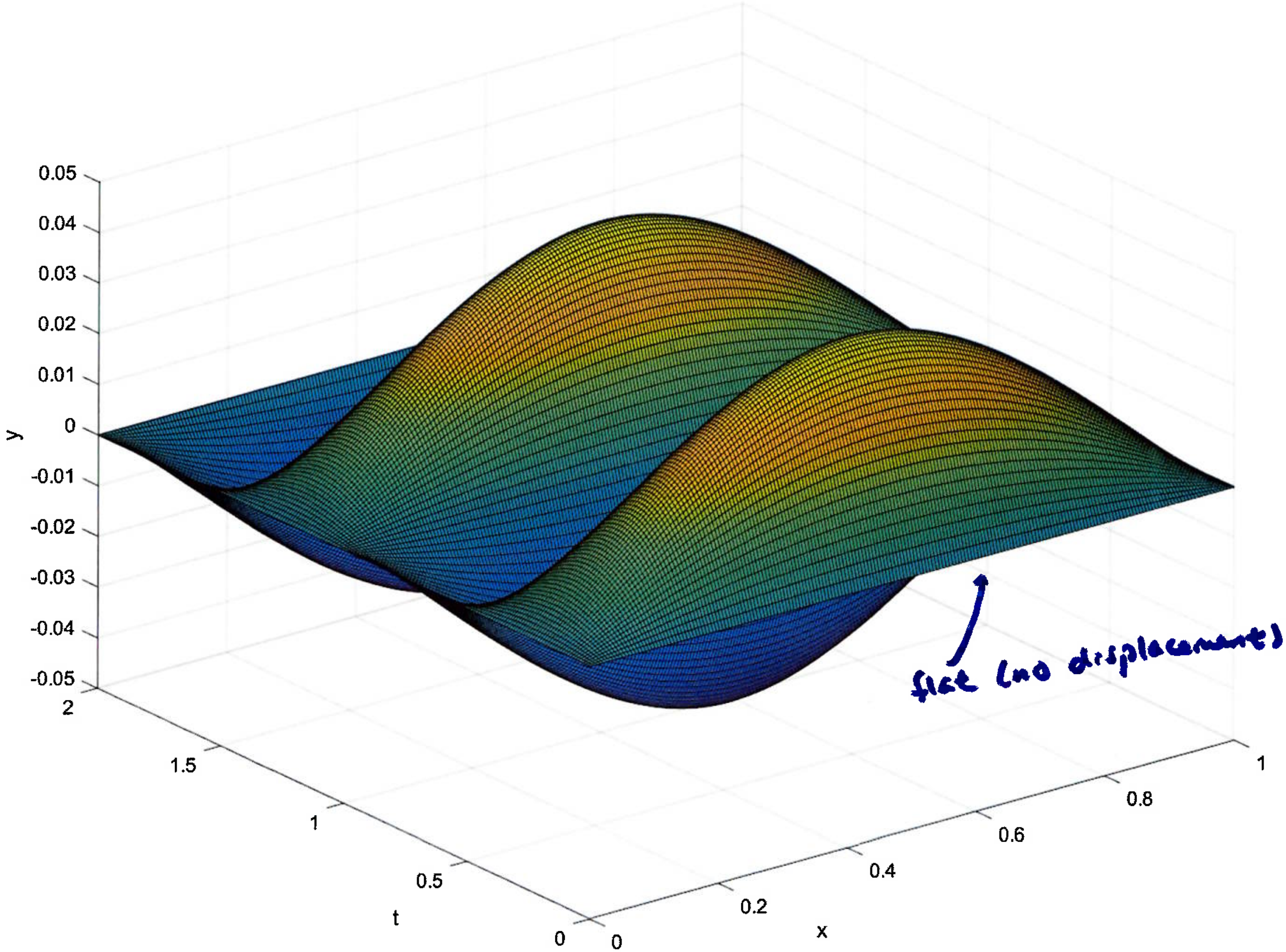
example: $y_{tt} = 4y_{xx}$ $\quad 0 < x < 1$ $\quad a=2,\ L=1$

$y(0,t) = y(L,t) = 0$

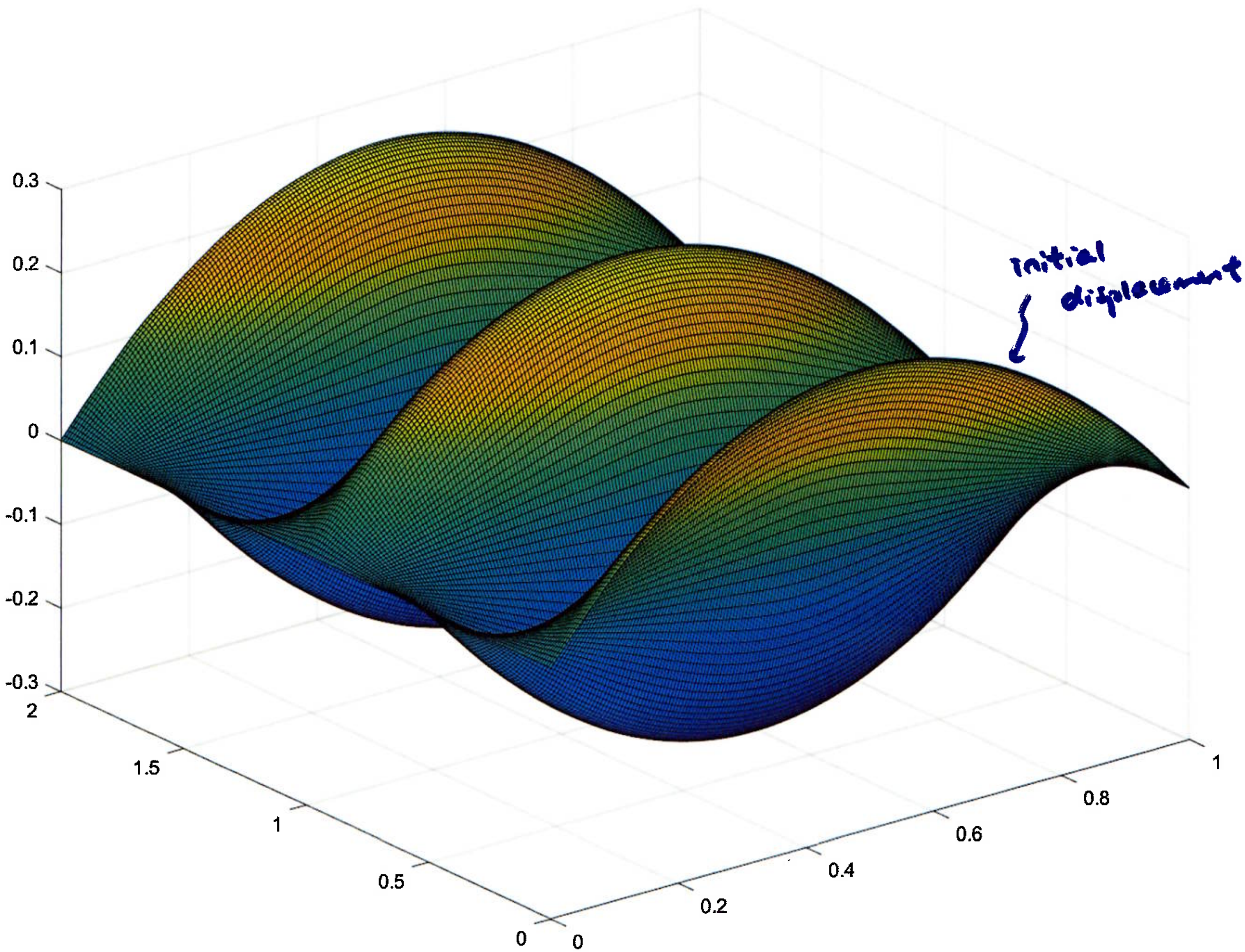
$y(x,0) = 0$

$y_t(x,0) = x(1-x)$





0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
-0.05
y
2
1.5
1
0.5
0
t
0
0.2
0.4
0.6
0.8
1
x
flat (no displacement)



0.3
0.2
0.1
0
-0.1
-0.2
-0.3
2
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1
Initial displacement

$$B_n = \frac{1}{n\pi}\int_0^1 x(1-x)\sin(n\pi x)\,dx$$

$$= \cdots = \frac{2}{n^4\pi^4}\left(1-(-1)^n\right) = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n^4\pi^4} & n \text{ odd} \end{cases}$$

Solution: $$y(x,t) = \sum_{n \text{ odd}}^{\infty} \frac{4}{n^4\pi^4} \underbrace{\sin(2n\pi t)}_{\text{time solution}} \underbrace{\sin(n\pi x)}_{\text{space solution}}$$

Compare to if $y(x,0) = x(1-x)$

$y_t(x,0) = 0$ (problem A)

$$y(x,t) = \sum_{n=1}^{\infty} A_n \cos(2n\pi t)\sin(n\pi x)$$

$$A_n = \frac{2}{1}\int_0^1 x(1-x)\sin(n\pi x)\,dx$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{8}{n^3\pi^3} & n \text{ odd} \end{cases}$$

$$y(x,t) = \sum_{n \text{ odd}}^{\infty} \frac{8}{n^3\pi^3}\cos(2n\pi t)\sin(n\pi x)$$

general case: non zero initial displacement and non zero initial velocity

the equation is linear: $y_{tt} = a^2 y_{xx}$

no dependent variable in coefficients

so the principle of superposition applies: general = prob A + prob B

$$y_{tt} = a^2 y_{xx} \qquad 0 < x < L$$

$$y(0,t) = y(L,t) = 0$$

$$y(x,0) = f(x)$$

$$y_t(x,0) = g(x)$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$y(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

animation showed two traveling waves

we can see this mathematically

Prob. A : $$y = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi a t}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

identity : $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

$$y = \frac{1}{2} \sum_{n=1}^{\infty} A_n \underbrace{\sin\left(\frac{n\pi}{L}(x+at)\right)}_{\text{wave that initially travels right}} + \frac{1}{2} \sum_{n=1}^{\infty} A_n \underbrace{\sin\left(\frac{n\pi}{L}(x-at)\right)}_{\text{wave to the left}}$$

D'Alembert solution (actually older than separation method)