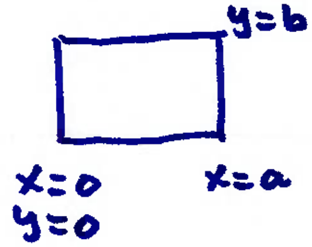
9.7 Steady-State Temperature and Laplace's Equation

1-D Heat eq: $u_t = k u_{xx}$

heat flows along x direction

2-D Heat eq: $u_t = k(u_{xx} + u_{yy})$



heat can flow in x and y direction

and higher dimensions

the right side is the Laplacien of u : $\nabla^2 u = \nabla \cdot \nabla \vec{u}$

Heat eq: $u_t = k \nabla^2 u$

Wave eq: $u_{tt} = a^2 \nabla^2 u$

Physical interpretation of $\nabla^2 u$: how the value of u at a point compares to the average value of u of all nearby points

for example, in 1-D Heat eq: $\nabla^2 u = u_{xx}$

$\nabla^2 u < 0$

concave down : u higher than avg

same idea if $\nabla^2 u > 0$

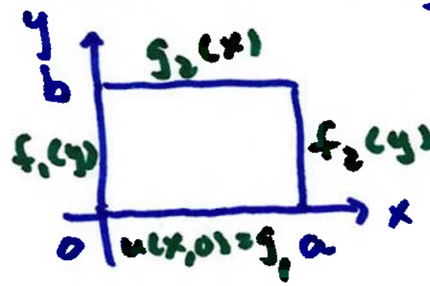
if heat and wave, $\nabla^2 u$ determines how heat flows (into/away)
or how string tries to resist.

heat/string don't "want" unevenness → $\nabla^2 u = 0$
→ happens when $u_t = 0$ → steady-state

let's look at the equation $\nabla^2 u = 0$

more specifically, $u_{xx} + u_{yy} = 0$ (steady-state of 2D heat eq).
↓
Laplace's Equation

region to consider: $0 < x < a \quad 0 < y < b$



$u_{xx} + u_{yy} = 0$ Four BC's: $u(x,0) = g_1(x)$

$u(0,y) = f_1(y)$

$u(a,y) = f_2(y)$

$u(x,b) = g_2(x)$

} if 0, then it's a homogeneous BC

this is a <u>linear</u> PDE, superposition applies

→ solve a case w/ 3 ~~ø~~ zero BC's, one non-zero BC

→ change which is non-zero (3 times)

→ combine all.

(all BC's = 0 leads to trivial solution $u=0$ which we don't care about) as an example, let's solve

$u_{xx} + u_{yy} = 0$

$u(0,y) = 0$ (left)

$u(x,0) = 0$ (bottom)

$u(x,b) = 0$ (top)

$u(a,y) = f(y)$ (right)

use separation of variables (useful for linear homogeneous PDE's)

$u(x,y) = X(x)Y(y)$ $\quad u_{xx} = X''Y$ $\quad u_{yy} = XY''$

$u(0,y) = X(0)Y(y) = 0 \longrightarrow X(0) = 0$

$u(x,0) = X(x)Y(0) = 0 \longrightarrow Y(0) = 0$

$u(x,b) = X(x)Y(b) = 0 \longrightarrow Y(b) = 0$

leave the nonhomogeneous BC last

$u_{xx} + u_{yy} = 0 \rightarrow X''Y + XY'' = 0$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \text{constant} = c$$

in Laplace's eq. we need more flexibility ($c<0$, $c=0$, $c>0$)

separate into two ODE's

$$\left.\begin{aligned} X'' - cX &= 0 \\ Y'' + cY &= 0 \end{aligned}\right\}$$

solve the one with complete BC's

here, $Y(0)=0$, $Y(b)=0$

$X(0)=0$

solve Y eq. first

$Y'' + cY = 0 \quad Y(0) = Y(b) = 0 \quad$ assume $c > 0$

$r^2 + c = 0 \quad r = \pm\sqrt{c}$

$$Y(y) = c_1 \cos(\sqrt{c}\, y) + c_2 \sin(\sqrt{c}\, y)$$

$Y(0) = 0 \rightarrow 0 = c_1$

$Y(b) = 0 \rightarrow 0 = c_2 \sin(\sqrt{c}\, b) \quad c_2 \neq 0$

$\sin(\sqrt{c}\, b) = 0$

$\sqrt{c}\, b = n\pi$

$$\boxed{\lambda_n = c = \frac{n^2\pi^2}{b^2}}$$

eigenvalues for $c > 0$

$$\boxed{Y_n = \sin\left(\frac{n\pi y}{b}\right)}$$

eigenfunctions

if we repeat w/ $c \le 0$, we found only trivial solutions
so there are no eigenvalues/functions for $c \le 0$ (not always the case)

now solve $X'' - cX = 0$

$$X'' - \frac{n^2\pi^2}{b^2} X = 0 \qquad r^2 - \frac{n^2\pi^2}{b^2} = 0 \qquad r = \pm \frac{n\pi}{b}$$

$$X(x) = c_1 e^{\frac{n\pi}{b}x} + c_2 e^{-\frac{n\pi}{b}x}$$

it's often more convenient to use $\sinh(x)$ and $\cosh(x)$
↳ $\frac{1}{2}(e^x - e^{-x})$

$$X(x) = d_1 \cosh\left(\frac{n\pi x}{b}\right) + d_2 \sinh\left(\frac{n\pi x}{b}\right)$$

now it's easier to use the 3rd BC : $X(0) = 0$

$$X(0) = d_1 = 0 \qquad \boxed{X_n = \sinh\left(\frac{n\pi x}{b}\right)}$$ eigenfunction for X

$u(x,y) = XY$

for each $n = 1, 2, 3, \ldots$ $\quad u_n = X_n Y_n = \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$

$u(x,y)$ is linear combo of all

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right)}$$

last (nonhomogeneous) BC: $u(a,y) = f(y)$ right edge

$$u(a,y) = f(y) = \sum_{n=1}^{\infty} \left[c_n \sinh\left(\frac{n\pi a}{b}\right)\right] \sin\left(\frac{n\pi y}{b}\right)$$

sine series with cs coefficients

coefficient of sine series

$$c_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b} \int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy$$

equivalent of L
(length in y direction)

$$\boxed{c_n = \frac{2}{b \sinh\left(\frac{n\pi a}{b}\right)} \int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy}$$