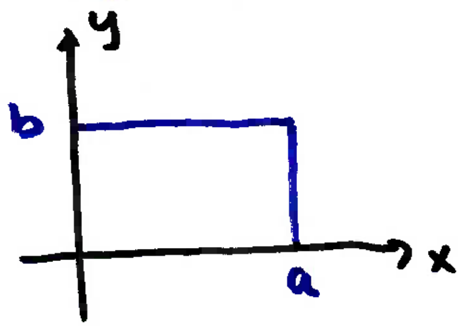
9.7 Laplace's Equation (part 2)

$u_{xx} + u_{yy} = 0 \qquad 0<x<a \qquad 0<y<b$



BC's: on four sides

$u(x,0) = 0$
$u(x,b) = 0$
$u(0,y) = 0$ } last time
$u(a,y) = f(x)$

let's try a different set of BC's

$u(x,0) = 0$ bottom held at zero

$u_x(0,y) = 0$
$u_x(a,y) = 0$ } insulate left and right

$u(x,b) = f(x)$ condition we decide

same procedure: separation of variables

$u(x,y) = X(x)Y(y) \qquad u_{xx} = X''Y \qquad u_{yy} = XY''$

$u_{xx} + u_{yy} = 0$

$X''Y + XY'' = 0 \rightarrow X''Y = -XY''$

$$\frac{X''}{X} = -\frac{Y''}{Y} = c$$

c depends on BC's: $u(x,0) = 0 \rightarrow X(x)Y(0) = 0 \rightarrow Y(0) = 0$

$u_x(0,y) = 0 \rightarrow X'(0)Y(y) = 0 \rightarrow X'(0) = 0$

$u_x(a,y) = 0 \rightarrow X'(a)Y(y) = 0 \rightarrow X'(a) = 0$

$X'' - cX = 0$ and $Y'' + cY = 0$

Solve X first because we have two BC's for X

if $c = 0$ $\quad X'' = 0 \rightarrow X(x) = C_1 x + C_2 \qquad X'(0) = X'(a) = 0$

$X'(x) = C_1 \longrightarrow C_1 = 0$

So, there is a nontrivial solution for $c = 0$

$$\boxed{X = 1, \quad c = \lambda = 0}$$

eigenvalue 0 eigenfunction 1

now check $c<0$, let $c=-\lambda^2$

$$X'' + \lambda^2 X = 0 \rightarrow X = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x) \qquad X'(0) = X'(a) = 0$$

$$X' = -\sqrt{\lambda} c_1 \sin(\sqrt{\lambda} x) + \sqrt{\lambda} c_2 \cos(\sqrt{\lambda} x)$$

same eigenvalues as last time

$$\boxed{\lambda_n = \frac{n^2\pi^2}{a^2} \qquad X_n = \cos\left(\frac{n\pi x}{a}\right)}$$

eigenvalues and eigenfunctions

$n = 0, 1, 2, 3, \ldots$

↑ covers the case when $c=0$

now solve for Y: $\frac{Y''}{Y} = -c \qquad Y'' + cY = 0$

$c=0$: $Y'' = 0 \rightarrow Y = c_1 y + c_2 \qquad Y(0) = 0$

$0 = c_2$

so $Y = c_1 y$

$$\boxed{Y = y, \qquad \lambda = 0}$$

eigenfunction eigenvalue

for $c<0$ (just like with X)

$$\boxed{Y_n = \sinh\left(\frac{n\pi y}{a}\right)}$$

for each $n = 0, 1, 2, 3, \dots$ $\quad u_n = X_n Y_n = (1)(y) = y$ when $n=0$

$$u_n = \sinh\left(\frac{n\pi y}{a}\right)\cos\left(\frac{n\pi x}{a}\right) \qquad n = 1, 2, 3, \dots$$

general solution is linear combo of all

$$\boxed{u(x,y) = \frac{1}{2}c_0 y + \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi y}{a}\right)\cos\left(\frac{n\pi x}{a}\right)}$$

so after applying last BC we get a cosine series

last BC: $u(x,b) = f(x)$

Combining with solution in the box

$$f(x) = \frac{1}{2}\underbrace{\left[c_0 b\right]}_{\text{"}a_0\text{"}} + \sum_{n=1}^{\infty} \underbrace{\left[c_n \sinh\left(\frac{n\pi b}{a}\right)\right]}_{\text{"}a_n\text{"}} \cos\left(\frac{n\pi x}{a}\right)$$

cosine series

$$C_0 \cdot b = \frac{2}{a}\int_0^a f(x)\underbrace{\cos\left(\frac{n\pi x}{a}\right)}_{=1} dx \qquad \boxed{C_0 = \frac{2}{ab}\int_0^a f(x)\underbrace{\cos\left(\frac{n\pi x}{a}\right)}_{1} dx}$$

"L" for x

$$C_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)dx \qquad \boxed{C_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)dx}$$

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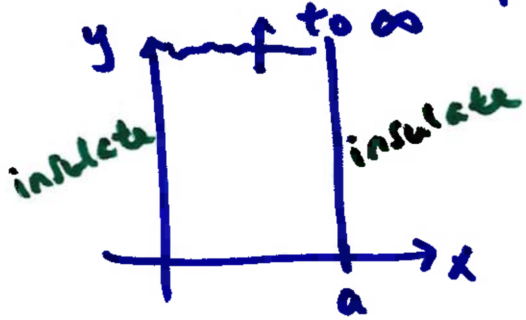
let's look a semi-infinite domain



$0 < x < a$

$y > 0$ (can go to infinity)

modeling a long strip of plate

insulate left and right: $u_x(0,y) = u_x(a,y) = 0$

bottom: $u(x,0) = f(x)$

"boundary condition" at ∞: temperature is bounded

$u = XY$ same X solution as 1st example

$$\boxed{X_n = \cos\left(\frac{n\pi x}{a}\right) \qquad \lambda_n = \frac{n^2\pi^2}{a^2} \qquad n = 0, 1, 2, 3, \ldots}$$

$$Y'' - \lambda Y = 0 \rightarrow Y'' - \frac{n^2\pi^2}{a^2} Y = 0 \rightarrow Y(y) = \underbrace{c_1 e^{\frac{n\pi}{a} y} + c_2 e^{-\frac{n\pi}{a} y}}_{\text{leave as exponential for the BC at } y=\infty}$$

BC at $y = \infty$ $\longrightarrow$ temp. is <u>bounded</u>

$c_1 = 0$ since $e^{\frac{n\pi y}{a}}$ goes to ∞ (unbounded) as $y \to \infty$

$$\boxed{Y_n = e^{-\frac{n\pi y}{a}}}$$

for each n, $u_n = X_n Y_n = e^{-\frac{n\pi y}{a}} \cos\left(\frac{n\pi x}{a}\right)$ $\quad n \neq 0$

$u_n = 1$ if $n = 0$

general solution:

$$\boxed{u(x,y) = \frac{1}{2} c_0 \cdot 1 + \sum_{n=1}^{\infty} c_n e^{-n\pi y/a} \cos\left(\frac{n\pi x}{a}\right)}$$

unused BC: $u(x,0) = f(x)$ (bottom)

$$f(x) = \frac{1}{2} C_0 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{a}\right)$$

$$\boxed{\begin{aligned} C_0 &= \frac{2}{a}\int_0^a f(x)\,dx \\ &\text{"L" in } x \\ C_n &= \frac{2}{a}\int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)dx \end{aligned}}$$