10.1 Sturm-Liouville Problem

in heat, wave, Laplace's we repeatedly encountered

$$X'' + \lambda X = 0 \qquad X(0) = X(L) = 0 \quad \text{(ends at 0 temp)}$$

$$\text{or} \qquad X'(0) = X'(L) = 0 \quad \text{(ends are insulated)}$$

in the 1st set of BC's, we ended up with

eigenvalues $\lambda_n = \dfrac{n^2\pi^2}{L^2}$

eigenfunctions $X_n = \sin\left(\dfrac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$

in the 2nd set, we got

eigenvalues $\lambda_n = \dfrac{n^2\pi^2}{L^2}$

eigenfunctions $X_n = \cos\left(\dfrac{n\pi x}{L}\right) \qquad n = 0, 1, 2, 3, \ldots$

both cases gave eigenvalues that are <u>nonnegative</u> and form

an increasing sequence $\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots$

and the eigenfunctions (one for eigenvalue) are <u>mutually orthogonal</u>

$$\int_0^L X_n X_m \, dx = 0 \quad \text{if } n \neq m$$

what if we change the BC's, for example

$$X(0) = X'(L) = 0 \qquad \text{(left frozen, right insulated)}$$

or

$$X(0) - X'(0) = 0 \qquad \text{(left temp = heat flux at left)}$$
$$X(L) + X'(L) = 0$$

or instead of $u_t = k u_{xx}$ we use $u_t = k(x) u_{xx}$

constant

conductivity is function of position

will the eigenvalues still be nonnegative and form increasing sequence?

will the eigenfunctions remain mutually orthogonal?

Sturm-Liouville Problem (SL)

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - q(x)y + \lambda r(x) y = 0 \qquad a<x<b$$

BC's: $\alpha_1 y(a) - \alpha_2 y'(a) = 0$. not both α_1 and α_2 can be zero

$\beta_1 y(b) + \beta_2 y'(b) = 0$ " " β_1 " β_2 " "

if $p'(x)$, $p(x)$, $q(x)$, and $r(x)$ are continuous on $[a, b]$

and $p(x) > 0$, $r(x) > 0$ on $[a, b]$

→ the SL problem is said to be regular

(most practical heat / wave covered)

if $q(x)$, α_1, α_2, β_1, β_2 are all nonnegative, then

the eigenvalues (λ) are also nonnegative (and real)

and $\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots$

and the eigenfunctions y_n are mutually ~~orgot~~ orthogonal with respect to the weight function $r(x)$

$$\int_a^b y_i y_j r(x)\,dx = 0$$

$y'' + \lambda y = 0 \qquad 0 < x < L$

$y(0) = 0 \qquad p(x) = 1,\ q(x) = 0,\ r(x) = 1$

$y(L) = 0 \qquad \alpha_2 = 0,\ \alpha_1 = 1,\ \beta_1 = 1,\ \beta_2 = 0$

so, this is a regular SL problem

$$\left.\begin{array}{l} \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \\ \displaystyle\int_0^L y_i y_j\,dx = 0 \end{array}\right\}$$

agrees with heat eq.

$\lambda = \frac{n^2\pi^2}{L^2}, \quad y_n = \sin\left(\frac{n\pi x}{L}\right)$

$y'' + \lambda y = 0 \qquad 0 < x < L$

$y(0) + y'(0) = 0 \qquad \alpha_1 = 1,\ \alpha_2 = -1 \rightarrow$ so, not regular

$y(L) = 0$

$\downarrow$

not guaranteed to have nonnegative λ's only

$y'' + \lambda y = 0 \qquad 0 < x < L$

$y(0) = 0 \qquad (\alpha_1 = 1, \ \alpha_2 = 0)$

$hy(L) - y'(L) = 0 \quad (h > 0) \qquad (\beta_1 = h > 0, \ \beta_2 = -1 < 0)$

$\downarrow$

not regular

eigenvalue could be negative

let's find the smallest eigenvalue (and the associated eigenfunction)

$y'' + \lambda y = 0$

Since it is not a regular SL problem, we start with finding negative eigenvalues $\lambda < 0$

for convenience, let $\lambda = -k^2$ (k is a positive constant)

$y'' - k^2 y = 0$

solution: $y = c_1 e^{kx} + c_2 e^{-kx}$

or $y = d_1 \cosh(kx) + d_2 \sinh(kx)$ $\leftarrow$ more convenient for BC's

$y(0)=0 \rightarrow \quad 0 = d_1 \cosh(0) + d_2 \sinh(0)$

$\qquad\qquad\qquad = d_1$

So, $\quad y = d_2 \sinh(Kx)\,, \quad y' = K \cdot d_2 \cosh(Kx)$

now $\quad h\,y(L) - y'(L) = 0$

$h\,d_2 \sinh(KL) - K d_2 \cosh(KL) = 0$

find out more about K because $\lambda = -K^2$

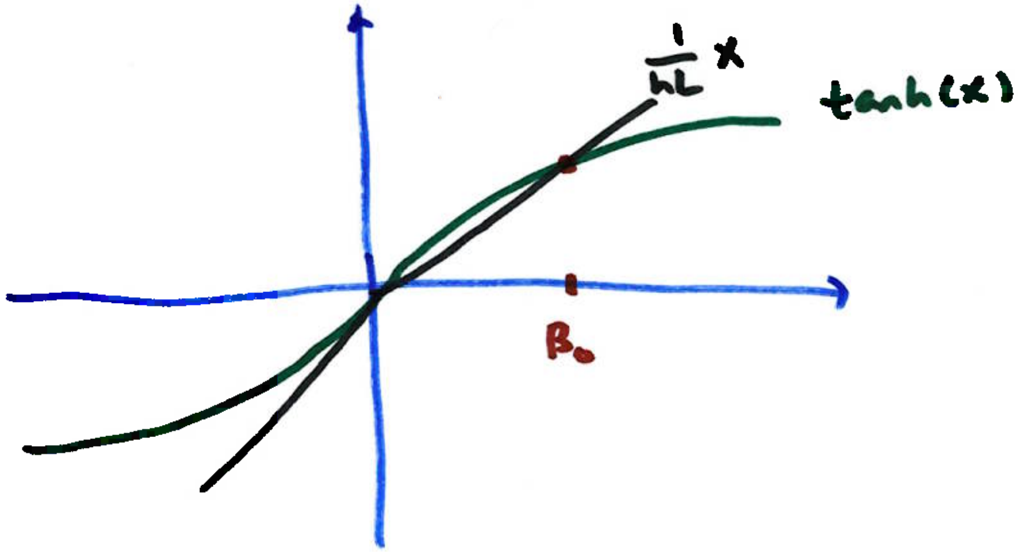
$d_2 \neq 0$

$h \sinh(KL) = K \cosh(KL)$

$$\tanh(KL) = \frac{K}{h} = \frac{KL}{h\cancel{K}}$$

$$\text{Solve } \tanh(KL) = \frac{KL}{hL} \longrightarrow \text{ Solve } \tanh(x) = \frac{x}{hL}$$

line w/ pos. slop



β_0 is the smallest positive intersection

$x = \beta_0 = KL$

$$\boxed{K = \frac{\beta_0}{L}}$$

smallest eigenvalue $\lambda = -k^2 = -\frac{\beta_0^2}{L^2}$

$\beta_0 \approx hL = kL \quad \rightarrow \quad k \approx h$, so $\lambda \approx -h^2$ (smallest eigenvalue)

eigenfunction: $y = d_2 \sinh(kx)$

associated w/ $\lambda = -h^2$

is $y = \sinh(hx)$

(next, $\lambda = 0$, $\lambda > 0$, collect all eigenfunctions, form solution)

will resemble a Fourier-like series