10.1 Sturm-Liouville Problem (continued)

from last time: $y'' + \lambda y = 0 \quad 0 < x < L$

$y(0) = 0$

$hy(L) - y'(L) = 0 \quad h > 0$

we found λ could be negative

for $\lambda < 0$, $\boxed{\lambda_0 = -\frac{\beta_0^2}{L^2}}$ β_0 is the positive intersection of $\tanh(x) = \frac{1}{hL}x$

$\boxed{y_0 = \sinh\left(\frac{\beta_0}{L}x\right)}$

those are the only eigenvalue and eigenfunction for $\lambda < 0$

now check if $\lambda = 0$ gives us nontrivial solution

$y'' + \lambda y = 0 \rightarrow y'' = 0 \rightarrow y = c_1 x + c_2$

$y(0) = 0 \rightarrow 0 = c_2 \rightarrow y = c_1 x$

$hy(L) - y'(L) = 0 \rightarrow hc_1 L - c_1 = 0 \quad c_1 \neq 0$

$hL = 1$

so, if $\boxed{\lambda = 0}$ the eigenfunction is $\boxed{x}$ $\rightarrow$ $\boxed{\begin{array}{l}\lambda_1 = 0 \\ y_1 = x\end{array}}$

now $\lambda > 0$, for convenience, let $\lambda = k^2$ $\quad k>0$

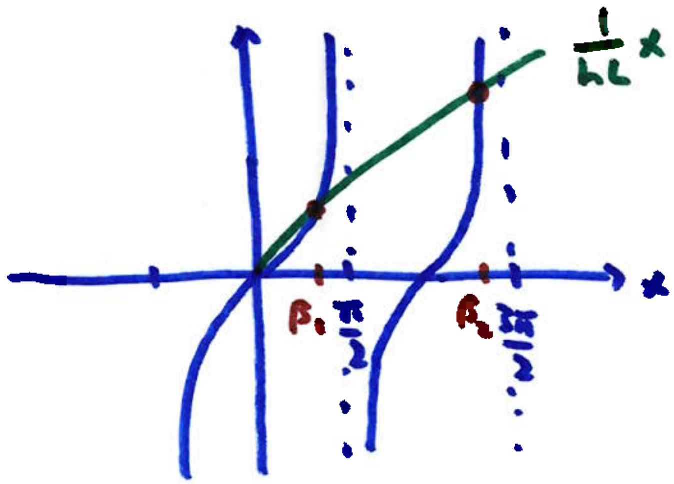
$$y'' + k^2 y = 0 \rightarrow y = c_1 \cos(kx) + c_2 \sin(kx)$$

$$y(0) = 0 \rightarrow 0 = c_1 \rightarrow y = c_2 \sin(kx)$$

$$hy(L) - y'(L) = 0 \rightarrow hc_2 \sin(kL) - c_2 k \cos(kL) = 0 \qquad c_2 \neq 0$$

$$h \sin(kL) = k \cos(kL)$$

$$\tan(kL) = \frac{k}{h} = \frac{kL}{hL} \rightarrow \text{solve } \tan(x) = \frac{1}{hL} x$$



$$\beta_n = k_n L = \sqrt{\lambda_n} L$$

$$\boxed{\lambda_n = \frac{\beta_n^2}{L^2}}$$

$$\boxed{y_n = \sin\left(\frac{\beta_n}{L} x\right)}$$

$\beta_n \rightarrow \frac{(2n-1)\pi}{2}$ as n increases

(y_n start to look like sine series)

all these eigenfunctions y_i are mutually orthogonal (just like in a standard Fourier series)

$$\int_0^L y_i y_j \, dx = 0 \qquad i \neq j$$

for example, $y'' + \lambda y = 0 \qquad 0 < x < \pi$

$y(0) = y(\pi) = 0$ (rod length L ends at 0 deg)

$\lambda_n = n^2, \quad y_n = \sin(nx)$

(stand Fourier Sine $\lambda = \frac{n^2\pi^2}{L^2}, \quad \sin\left(\frac{n\pi x}{L}\right)$)

$$\int_0^\pi y_n y_m \, dx = \int_0^\pi \sin(nx)\sin(mx)\,dx = 0 \quad n \neq m$$

$$\int_0^\pi \sin(nx)\sin(nx)\,dx = \frac{\pi}{2} \quad \left(\frac{L}{2}\right)$$

Sine series: $f(x) = b_1 \sin(x) + b_2 \sin(2x) + \cdots$

if we want b_3.

$$f(x)\sin(3x) = b_1\sin(x)\sin(3x) + b_2\sin(2x)\sin(3x) + b_3\sin(3x)\sin(3x) + \dots$$

$$\int_0^\pi f(x)\sin(3x)\,dx = \cancelto{0}{\int_0^\pi b_1\sin(x)\sin(3x)\,dx} + \int_0^\pi b_2\sin(2x)\sin(3x)\,dx + \cancelto{0}{\dots}$$

$$\int_0^\pi f(x)\sin(3x)\,dx = b_2\cdot\frac{\pi}{2}$$

$$b_3 = \frac{1}{\pi/2}\int_0^\pi f(x)\sin(3x)\,dx \to b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx$$

L

we can represent many functions using mutually orthogonal eigenfunctions (Fourier series, for example)

process is the same, but coefficients will be more complicated-looking (not $\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$, for example)

for example, $y'' + \lambda y = 0 \quad 0 < x < 3$

$y(0) = 0$

$y(3) + y'(3) = 0$

it is a regular SL problem, $\lambda \geq 0$

if we solved it, we get $\lambda_n = \frac{\beta_n^2}{9}$ β_n are intersections of $\tan(x) = -\frac{1}{9}x$

$\beta = 2.8, 5.7, 8.7, 11.7, \ldots$

$y_n = \sin\left(\frac{\beta_n}{3}x\right)$

↳ not integer so not standard Fourier

suppose we want to expand $f(x) = 1$ using these y_n,

$$1 = c_1 y_1 + c_2 y_2 + c_3 y_3 + \cdots + c_n y_n + \cdots \qquad y_n = \sin\left(\frac{\beta_n}{3} x\right)$$

what do the c_n's look like?

$$\int_0^3 y_i y_j \, dx = 0 \qquad \int_0^3 y_i y_i \, dx \neq 0$$

$$1 \cdot y_n = c_1 y_1 y_n + c_2 y_2 y_n + \cdots$$

integrate

$$\int_0^3 1 \cdot y_n \, dx = \int_0^3 c_1 \cancelto{0}{y_1 y_n} \, dx + \int_0^3 c_2 \cancel{y_2 y_n} \, dx$$

$$+ \cancelto{0}{\cdots} + \int_0^3 c_n \underbrace{y_n y_n}_{\neq 0} \, dx + \int_0^3 \cancelto{0}{c_{n+1} y_{n+1} y_n} \, dx + \cancelto{0}{\cdots}$$

$$c_n = \frac{\int_0^3 1 \cdot y_n \, dx}{\int_0^3 y_n y_n \, dx} = \cdots = \frac{2 - 2\cos(\beta_n)}{\beta_n - \sin(\beta_n)\cos(\beta_n)}$$

sub in $y_n = \sin\left(\frac{\beta_n}{3} x\right)$