7.1 Laplace Transform

a type of integral transform (a related one is the Fourier Transform)

Laplace Transform of $f(t)$ is defined

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, \underbrace{e^{-st}}_{\text{kernel of transformation}}\, dt = F(s)$$

example $f(t) = 1$

$$\mathcal{L}\{1\} = \int_0^\infty 1 \cdot e^{-st}\, d\underline{\underline{t}}$$

t is variable
treat S as constant

$$= \lim_{a\to\infty} \int_0^a e^{-st}\, dt = \lim_{a\to\infty} \frac{1}{-s} e^{-st} \Big|_{t=0}^{t=a}$$

$$= \lim_{a\to\infty} \left(\frac{1}{-s} e^{-s\cdot a} - \frac{1}{-s} e^{-s\cdot 0} \right)$$

s must be positive for integral to converge $\rightarrow$ $s > 0$

$$= \lim_{a\to\infty} \left(0 + \frac{1}{s} \right) = \boxed{\frac{1}{s} \quad , s>0}$$

example $f(t) = t$

LATE

log algebraic exponential trig

$\uparrow$ t

$$\mathcal{L}\{t\} = \int_0^\infty t \cdot e^{-st}\,dt = \lim_{a\to\infty} \int_0^a t \cdot e^{-st}\,dt$$

by parts

$u = t \qquad dv = e^{-st}\,dt$

$du = dt \qquad v = -\frac{1}{s} e^{-st}$

$$= \lim_{a\to\infty} \left(uv - \int v\,du \right)$$

$$= \lim_{a\to\infty} \left(-\frac{t}{s} e^{-st} \Big|_{t=0}^{t=a} + \int_0^a \frac{1}{s} e^{-st}\,dt \right)$$

$$= \lim_{a\to\infty}\left(-\frac{t}{s}e^{-st}\Big|_{t=0}^{t=a} - \frac{1}{s^2}e^{-st}\Big|_{t=0}^{t=a} \right)$$

$$= \lim_{a\to\infty}\left(\underbrace{-\frac{a}{s}e^{-sa}} + 0 - \underbrace{\frac{1}{s^2}e^{-sa}} + \frac{1}{s^2} \right)$$

go away if $s>0$

$$= \frac{1}{s^2}$$

$$\boxed{\mathcal{L}\{t\} = \frac{1}{s^2}, \quad s>0}$$

$$\mathcal{L}\{c_1 + c_2 t\} = \int_0^\infty (c_1 + c_2 t)e^{-st}\,dt$$

$$= \int_0^\infty c_1 e^{-st}\,dt + \int_0^\infty c_2 t e^{-st}\,dt$$

$$= c_1 \cdot \underbrace{\int_0^\infty 1 \cdot e^{-st}\,dt}_{\mathcal{L}\{1\}} + c_2 \cdot \underbrace{\int_0^\infty t e^{-st}\,dt}_{\mathcal{L}\{t\}} = c_1\mathcal{L}\{1\} + c_2\mathcal{L}\{t\}$$

this shows the Laplace transform is <u>linear</u>

because integration is linear

so, in general, $\mathcal{L}\{c_1 f(t) + c_2 g(t)\}$

$$= c_1 \mathcal{L}\{f(t)\} + c_2 \mathcal{L}\{g(t)\}$$

$$= c_1 F(s) + c_2 G(s)$$

provided that those exist

sum/diff → sum/diff. of LT's

BUT, $\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$

given $f(t)$, what are the requirements for the existence of $\mathcal{L}\{f(t)\} = F(s)$?

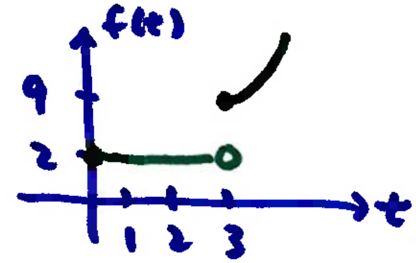
$$\mathcal{L}\{f(t)\} = \underbrace{\int_0^\infty f(t) e^{-st}\, dt}$$

as long as this exists, $\mathcal{L}\{f(t)\}$ exists

→ as long as $f(t)$ is piecewise continuous on $0 < t < \infty$ (finite # of discontinuities)

Example

$$f(t) = \begin{cases} 0 & t < 0 \\ 2 & 0 \le t < 3 \\ t^2 & t \ge 3 \end{cases}$$



$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \int_0^3 2e^{-st}\, dt + \int_3^\infty t^2 e^{-st}\, dt$$

$$\vdots$$

$$= \boxed{\frac{2-2e^{-3s}}{s} + \frac{e^{-3s}(3s+1)}{s^2}, \quad (s>0)}$$

in practice, we don't normally do LT by hand

often, we refer to a table of common LT's

<u>example</u> $f(t) = 3t^4 - e^{5t} + 2\sin 3t$

$$F(s) = 3\mathcal{L}\{t^4\} - \mathcal{L}\{e^{5t}\} + 2\mathcal{L}\{\sin 3t\}$$

table lookups

$$= 3 \cdot \frac{4!}{s^{4+1}} - \frac{1}{s-5} + 2 \cdot \frac{3}{s^2+3^2} \quad (s > 5)$$

inverse LT : $f(t) = \mathcal{L}^{-1}\{F(s)\}$

$$F(s) = \frac{1}{s}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

table look up
under F(s) column

$f(t)$	$F(s)$	
1	$\frac{1}{s}$	$(s > 0)$
t	$\frac{1}{s^2}$	$(s > 0)$
$t^n \quad (n \geqq 0)$	$\frac{n!}{s^{n+1}}$	$(s > 0)$
$t^a \quad (a > -1)$	$\frac{\Gamma(a+1)}{s^{a+1}}$	$(s > 0)$
e^{at}	$\frac{1}{s-a}$	$(s > a)$
$\cos kt$	$\frac{s}{s^2+k^2}$	$(s > 0)$
$\sin kt$	$\frac{k}{s^2+k^2}$	$(s > 0)$
$\cosh kt$	$\frac{s}{s^2-k^2}$	$(s > \lvert k \rvert)$
$\sinh kt$	$\frac{k}{s^2-k^2}$	$(s > \lvert k \rvert)$
$u(t-a)$	$\frac{e^{-as}}{s}$	$(s > 0)$