Recap (ch. 5, ch. 6)

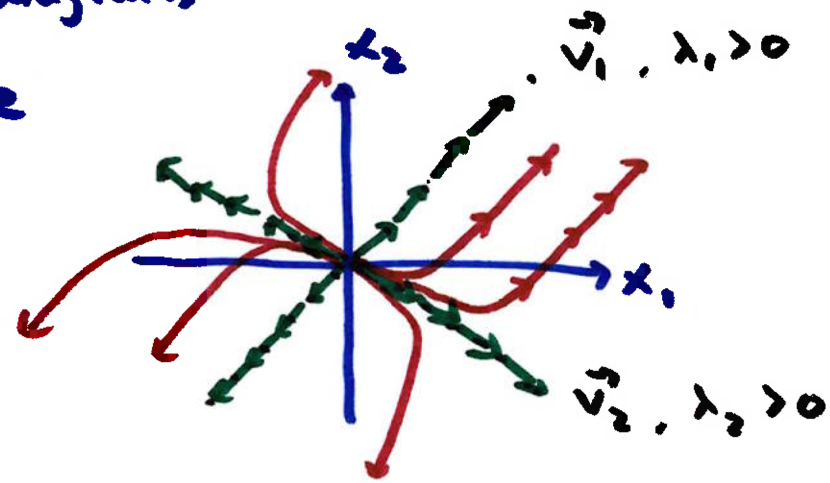
$\vec{x}' = A\vec{x}$ solution: $\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \cdots$

$\lambda_i, \vec{v}_i$ are the eigenvalue/eigenvector pairs

focus on 2D, phase diagrams

$\lambda_1 > \lambda_2$, both positive

follow this eigenvector when t is large

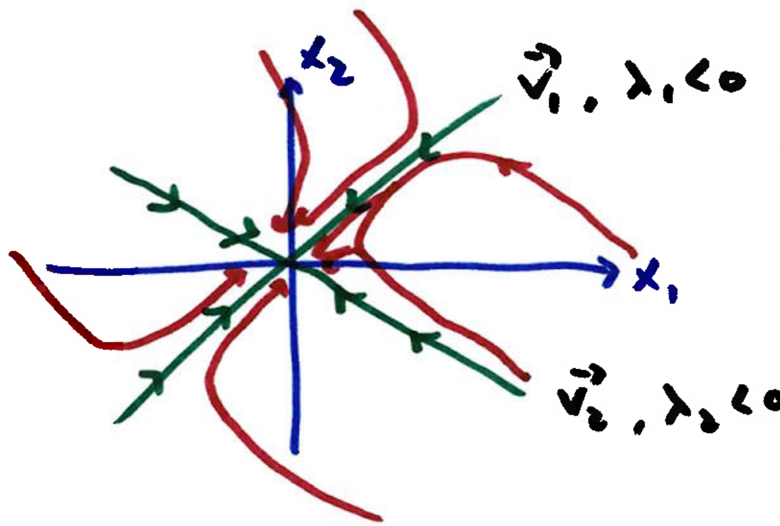


nodal source

unstable

both λ's < 0, $\lambda_1 > \lambda_2$

follow this eigenvector close to origin



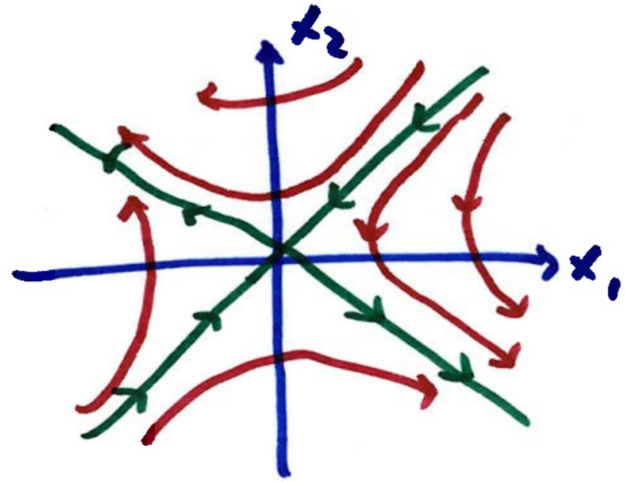
nodal sink

asymp. stable

qualitatively, if $\lambda_1 = \lambda_2$, same picture ~~both~~ but w/o two straight line solutions

unless there are two eigenvectors then star source/sink (rare)

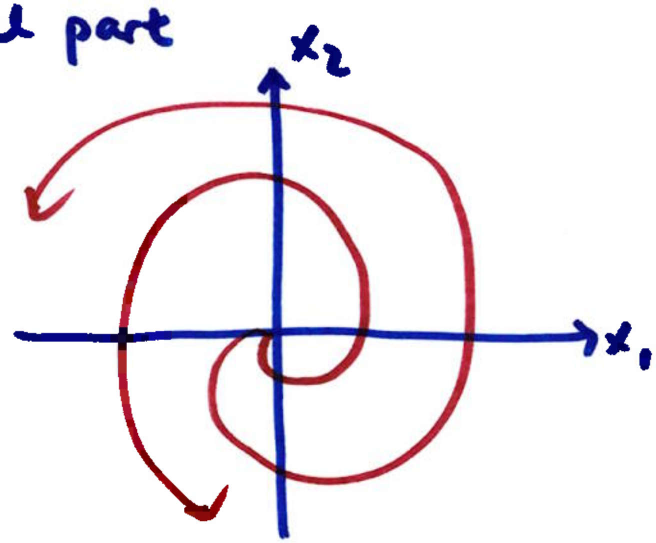
λ's opposite signs



origin is saddle point

unstable

λ's complex, positive real part

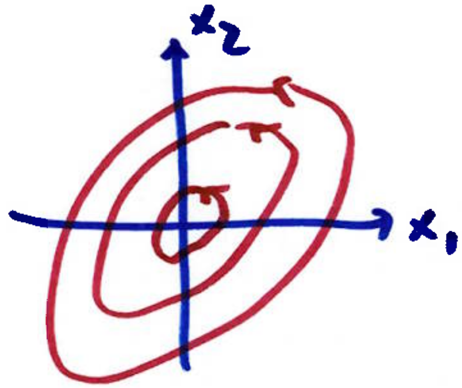


spiral source

unstable

if negative real part, spiral sink <u>asymp. stable</u>

real part zero



center

<u>stable</u>

nonlinear systems

$$\frac{dx}{dt} = f(x,y)$$

$$\frac{dy}{dt} = g(x,y)$$

critical points : equilibrium solutions that make x' and y' both zero $\rightarrow f = g = 0$

linearize system about the critical points

Jacobian : $J(x_0, y_0) = \begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix}$

near cp: $\begin{bmatrix} u' \\ v' \end{bmatrix} = J(x_0, y_0) \begin{bmatrix} u \\ v \end{bmatrix}$ $u = x - x_0$, $v = y - y_0$

the eigenvalue/eigenvector pairs of J tell us the type of equilibrium that cp is.

exceptions: J has eigenvalues of opposite signs (saddle)

" " " that are purely imaginary (center)

→ linearized system may or may not tell us the true behavior

Predator-Prey

x : prey

y : predator

$$\frac{dx}{dt} = ax - pxy$$

$$\frac{dy}{dt} = -by + qxy$$

exponential growth w/o predator

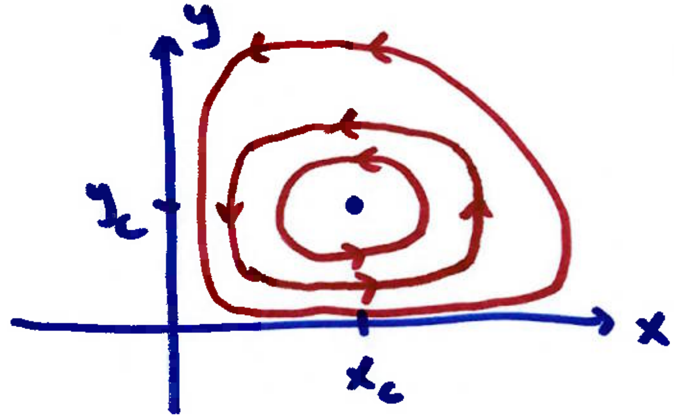
predator reduces growth rate

prey increases growth rate

exponential decay w/o prey

$a, b, p, q > 0$

typical behavior



$(0,0)$ unstable

(x_c, y_c) stable

Competitors

two fight for same resource

$$\frac{dx}{dt} = \underbrace{a_1 x}_{\text{exp. growth}} - \underbrace{b_1 x^2}_{\text{logistic}} - \underbrace{c_1 xy}_{\text{interaction (further reduction of growth rates)}}$$

$$\frac{dy}{dt} = a_2 y - b_2 y^2 - c_2 xy$$

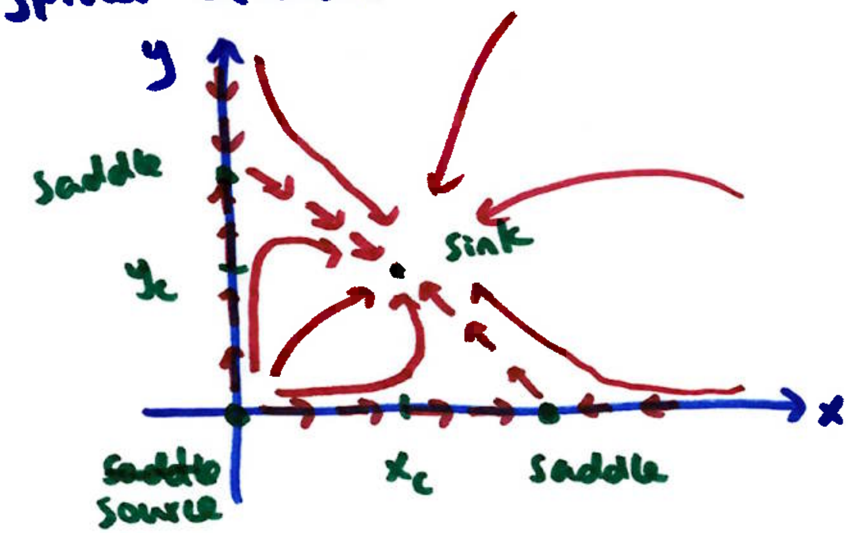
$$a_i, b_i, c_i > 0$$

$$\text{cp: } \underbrace{(0,0)}_{\text{all go extinct}},\ \underbrace{\left(\frac{a_1}{b_1}, 0\right),\ \left(0, \frac{a_2}{b_2}\right)}_{\text{one goes extinct one logistic growth}},\ \underbrace{(x_c, y_c)}_{\text{coexist}}$$

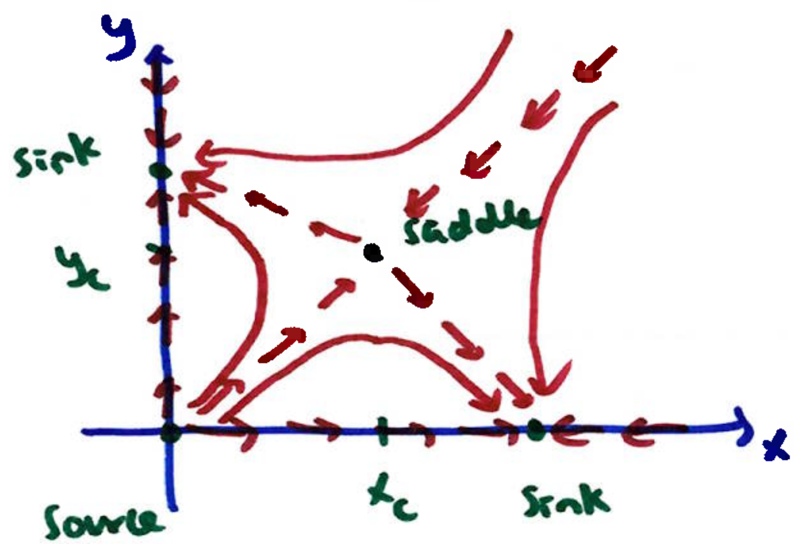
$b_1 b_2 > c_1 c_2$ intrinsic limits more important than competition

"weak" competition

typical behavior



$b_1 b_2 < c_1 c_2$ "strong" competition



Cooperation

$$\frac{dx}{dt} = a_1 x - b_1 x^2 - c_1 xy$$

$$\frac{dy}{dt} = a_2 y - b_2 y^2 - c_2 xy$$

$c_1, c_2 < 0$
(boost to x', y')

cp: $(0,0)$, $\left(\frac{a_1}{b_1}, 0\right)$, $\left(0, \frac{a_2}{b_2}\right)$, (x_c, y_c)

$x_c > \frac{a_1}{b_1}$

$y_c > \frac{a_2}{b_2}$