Laplace Transform

$y'' + y = 3 \qquad y(0) = 2 \qquad y'(0) = 1$

$\mathcal{L}\{y\}$

$\mathcal{L}\{y'\} = sY - y(0)$

$\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0)$

$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{3\}$

$s^2Y - sy(0) - y'(0) + Y = \frac{3}{s}$ find Y, then $y = \mathcal{L}^{-1}\{Y\}$

$(s^2+1)Y = 2s + 1 + \frac{3}{s}$

$Y = \frac{2s}{s^2+1} + \frac{1}{s^2+1} + \frac{3}{s(s^2+1)}$

$= \frac{2s}{s^2+1} + \frac{1}{s^2+1} + \frac{3}{s} - \frac{3s}{s^2+1}$

$= \frac{3}{s} + \frac{1}{s^2+1} - \frac{s}{s^2+1}$

$y = 3 + \sin(t) - \cos(t)$

$\frac{3}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$

$3 = A(s^2+1) + (Bs+C)(s)$

$3 = (A+B)s^2 + Cs + A$

$A+B=0$

$C=0$

$A=3$

$B=-3$

convolution: $f(t) * g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$

$$\mathcal{L}\{f(t) * g(t)\} = F(s)\, G(s)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\,\frac{1}{s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2}\,\frac{1}{s-1} - \frac{1}{2}\,\frac{1}{s+1}\right\}$$

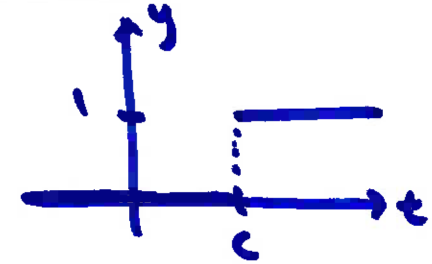
$$\mathcal{L}^{-1}\left\{\overset{F}{\boxed{\frac{1}{s-1}}}\ \overset{G}{\boxed{\frac{1}{s+1}}}\right\} \qquad f(t) = e^{t} \qquad g(t) = e^{-t}$$

$$= \int_0^t e^{t-\tau} e^{-\tau}\, d\tau = \int_0^t e^{t} e^{-2\tau}\, d\tau = e^{t} \int_0^t e^{-2\tau}\, d\tau$$

$$= e^{t}\left(-\frac{1}{2} e^{-2\tau}\right|_0^t = -\frac{1}{2} e^{t}\left(e^{-2t} - 1\right) = \frac{1}{2} e^{t} - \frac{1}{2} e^{-t}$$

$$= \sinh(t)$$

$$u(t-c) = \begin{cases} 1 & t \geq c \\ 0 & t < c \end{cases}$$



$$\mathcal{L}\{u(t-c)\} = \frac{e^{-cs}}{s}$$

$$\mathcal{L}\{u(t-c)\, f(t-c)\} = e^{-cs} F(s)$$

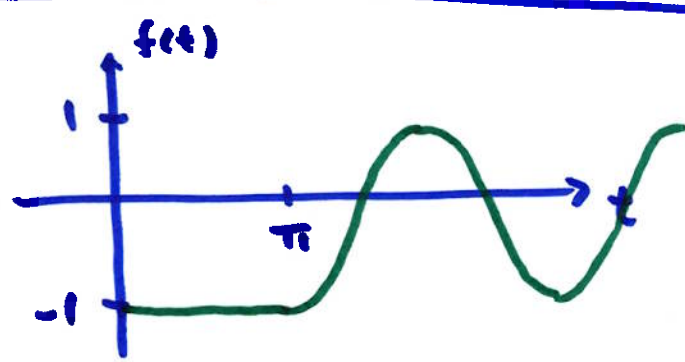
shift LEFT by c then transform

we don't transform the function as-is

$$\mathcal{L}^{-1}\{e^{-cs} F(s)\} = u(t-c)\, f(t-c)$$

shift RIGHT after inverse transform

$$f(t) = \begin{cases} -1 & 0 < t < \pi \\ \cos(t) & t \geq \pi \end{cases} \quad (t > 0)$$



write f(t) in terms of unit step

$$f(t) = -1 + u(t-\pi)\left[1 + \cos(t)\right]$$

$$F(s) = \mathcal{L}\{-1\} + \mathcal{L}\{u(t-\pi)[\underbrace{1+\cos(t)}]\}$$

Shift LEFT by π : $t \to t+\pi$

so we transform $1+\cos(t+\pi)$

$$= -\frac{1}{s} + e^{-\pi s}\mathcal{L}\{1+\cos(t+\pi)\}$$

$\cos(t+\pi) = -\cos(t)$

$$= -\frac{1}{s} + e^{-\pi s}\mathcal{L}\{1-\cos(t)\}$$

$$= -\frac{1}{s} + e^{-\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$$

$$F(s) = \frac{e^{-2s}}{s^2-2s+2} \qquad f(t) = ?$$

$$= e^{-2s}\boxed{\frac{1}{s^2-2s+2}}$$

inv. transform

then shift t RIGHT by 2 : $t \to t-2$

$$\frac{1}{s^2-2s+2} = \frac{1}{(s^2-2s+1)+1} = \frac{1}{(s-1)^2+1} \rightarrow e^t \sin(t)$$

$$f(t) = u(t-2)\left[e^{t-2}\sin(t-2)\right]$$

impulse : $\delta(t-c) = \begin{cases} \rightarrow\infty & t=c \\ 0 & t\neq c \end{cases}$ $\quad \int_{-\infty}^{\infty} \delta(t-c)\,dt = 1$

$$\mathcal{L}\{\delta(t-c)\} = e^{-cs}$$

$y'' + y = \delta(t-\pi) + \delta(t-2\pi) \qquad y(0)=0,\ y'(0)=0$

$$s^2Y - s\cancel{y(0)} - \cancel{y'(0)} + Y = e^{-\pi s} + e^{-2\pi s}$$

$$(s^2+1)Y = e^{-\pi s} + e^{-2\pi s}$$

$$Y = e^{-\pi s}\frac{1}{s^2+1} + e^{-2\pi s}\frac{1}{s^2+1} \qquad \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t)$$

$$y = u(t-\pi)\sin(t-\pi) + u(t-2\pi)\sin(t-2\pi)$$

$$= -u(t-\pi)\sin(t) + u(t-2\pi)\sin(t) = \begin{cases} 0 & 0<t<\pi \\ -\sin(t) & \pi<t<2\pi \\ 0 & t>2\pi \end{cases}$$