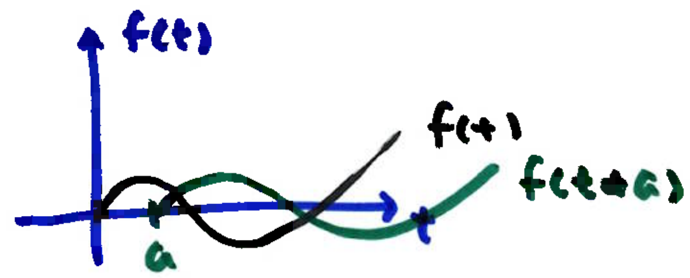
7.3 Translation of Laplace Transforms

given $f(t)$, $f(t-a)$ translates $f(t)$ by a units



$\mathcal{L}\{f(t)\} = F(s)$ what is $F(s-a)$? unfortunately, not as in time domain (t)

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt = F(s)$$

change s to $s-a$: $F(s-a) = \int_0^\infty f(t)e^{-(s-a)t}\,dt$

$$F(s-a) = \int_0^\infty f(t)e^{-st}e^{at}\,dt$$

$$= \int_0^\infty \left(f(t)e^{at}\right)e^{-st}\,dt = \boxed{\mathcal{L}\{f(t)e^{at}\} = F(s-a)}$$

example $\mathcal{L}\{\underbrace{e^{t}}_{a=1}\underbrace{\cos 2t}_{f(t)}\}$ $\qquad F(s-a)=\mathcal{L}\{e^{at}f(t)\}$

$f(t)=\cos 2t \qquad F(s)=\dfrac{s}{s^2+4}$

$$F(s-a)=F(s-1)=\frac{s-1}{(s-1)^2+4}$$

so, $\boxed{\mathcal{L}\{e^{t}\cos 2t\}=\dfrac{s-1}{(s-1)^2+4}}$

example $\mathcal{L}^{-1}\left\{\dfrac{3s+5}{\underbrace{s^2-6s+25}_{\text{sum/diff. of squares}}}\right\}$

$$\frac{3s+5}{s^2-6s+9+16}=\frac{3s+5}{(s-3)^2+4^2}$$

$$\frac{3(s-3)}{(s-3)^2+4^2} + \frac{14}{(s-3)^2+4^2}$$

$$= 3 \cdot \underbrace{\frac{s-3}{(s-3)^2+4^2}}_{\text{looks like}} + \frac{14}{4} \cdot \underbrace{\frac{4}{(s-3)^2+4^2}}_{\text{looks like}}$$

$\frac{s}{s^2+4^2} \to \cos 4t$ w/ s replaced by $\underbrace{s-3}_{a=3}$

$\cancel{\frac{4}{(s-3)^2+4^2}}$ $\frac{4}{s^2+4^2} \to \sin 4t$ w/ s replaced by $\underbrace{s-3}_{a=3}$

So, inverse Laplace of this gives

$$\boxed{3e^{3t}\cos 4t + \frac{7}{2}e^{3t}\sin 4t}$$

we could also just use table entries

$$\mathcal{L}\{e^{at}\cos kt\} = \frac{s-a}{(s-a)^2+k^2}$$

$$\mathcal{L}\{e^{at}\sin kt\} = \frac{k}{(s-a)^2+k^2}$$

$$\frac{3s+5}{s^2-6s+25} = \frac{3s}{(s-3)^2+4^2} + \frac{5}{(s-3)^2+4^2}$$

$$= \frac{3s-9}{(s-3)^2+4^2} + \frac{5+9}{(s-3)^2+4^2}$$

$$= 3\cdot\underbrace{\frac{s-3}{(s-3)^2+4^2}}_{e^{3t}\cos 4t} + \frac{14}{4}\cdot\underbrace{\frac{4}{(s-3)^2+4^2}}_{e^{3t}\sin 4t}$$

__example__ $x'' + 4x' + 13x = te^{-t} \qquad x(0) = 0, \; x'(0) = 2$

old way: solve homogeneous part $\quad x'' + 4x' + 13x = 0$

$$x = c_1(\cdots) + c_2(\cdots)$$

then solve for c_1, c_2 using initial conditions

then nonhomogeneous part

undetermined coeffs: $x_p = Ate^{-t} + Be^{-t}$

then plug into DE to solve for A, B

Laplace way: transform, solve for $\mathcal{L}\{x\}$, then $\mathcal{L}^{-1}$ that

$$\mathcal{L}\{x''\} + 4\mathcal{L}\{x'\} + 13\mathcal{L}\{x\} = \mathcal{L}\{te^{-t}\}$$

$$\left(s^2 X - \cancel{s x(0)}^{\,0} - x'(0)\right) + 4\left(s X - \cancel{x(0)}^{\,0}\right) + 13 X = \frac{1}{(s+1)^2}$$

$$(s^2 + 4s + 13) X - 2 = \frac{1}{(s+1)^2}$$

$$X = \frac{2}{\underbrace{s^2 + 4s + 13}_{(s+2)^2 + 3^2}} + \frac{1}{\underbrace{(s+1)^2 (s^2 + 4s + 13)}_{\text{partial fraction expansion}}}$$

$$X = \frac{2}{(s+2)^2+3^2} + \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+4s+13}$$

$$= \frac{2}{(s+2)^2+3^2} + \frac{1}{50}\cdot\frac{s-2}{(s+2)^2+3^2} - \frac{1}{50}\,\frac{1}{s+1} + \frac{1}{10}\,\frac{1}{(s+1)^2}$$

$$= \frac{2}{(s+2)^2+3^2} + \frac{1}{50}\left[\frac{s+2}{(s+2)^2+3^2} - \frac{4}{(s+2)^2+3^2}\right] - \frac{1}{50}\,\frac{1}{s+1} + \frac{1}{10}\,\frac{1}{(s+1)^2}$$

$$\boxed{x(t) = \frac{2}{3}e^{-2t}\sin(3t) + \frac{1}{50}\left[e^{-2t}\cos(3t) - \frac{4}{3}e^{-2t}\sin(3t)\right] - \frac{1}{50}e^{-t} + \frac{1}{10}te^{-t}}$$