9.2 General Fourier Series and Convergence (part 2)

$f(t)$ period $2L$ for $-L<t<L$ has Fourier series

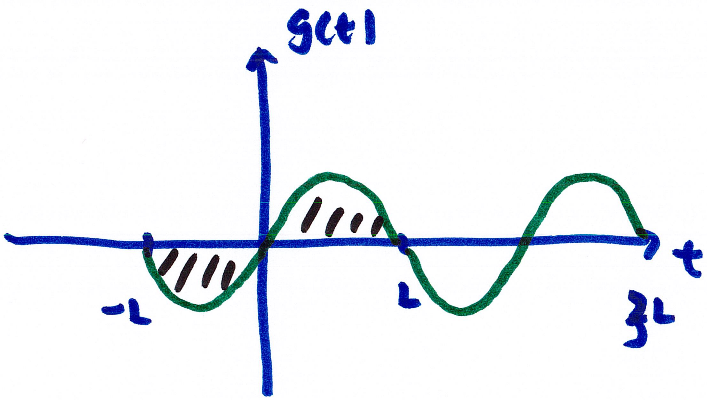
$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi t}{L}\right) + b_n\sin\left(\frac{n\pi t}{L}\right)\right]$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

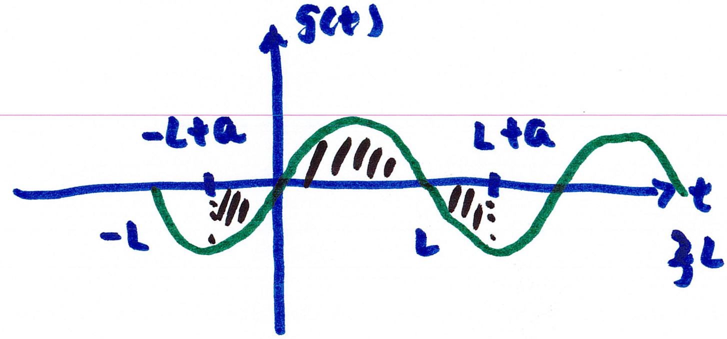
in practice, we usually start at $t=0$

so we want to modify the above for $f(t)$ period $2L$ for $0<t<2L$

if $g(t)$ is periodic w/ period $2L$, then $g(t+2L) = g(t)$

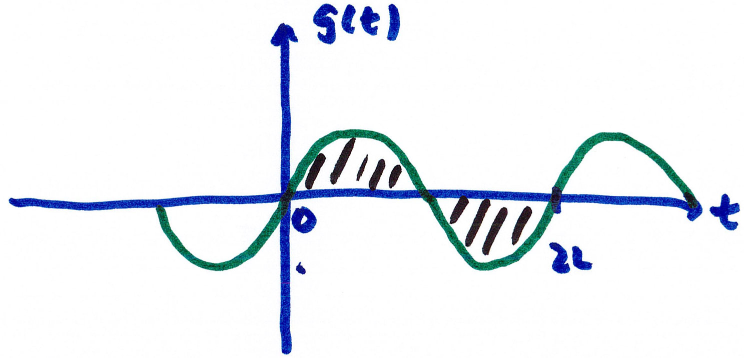


$\int_{-L}^{L} g(t)\,dt$ is the area of shaded region



$\int_{-L+a}^{L+a} g(t)\,dt$ is area of shaded region (same as before)

so, if $a = L$



$\int_{0}^{2L} g(t)\,dt$ = same as before

therefore, as long as the length of interval = period, integral remains the same

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

period 2L

period is $\frac{2\pi}{\frac{n\pi}{L}} = \frac{2L}{n}$

since multiples of period is also period

so $n\left(\frac{2L}{n}\right) = 2L$ so entire thing has period 2L so, we can shift integration interval as long as its length is $2L$

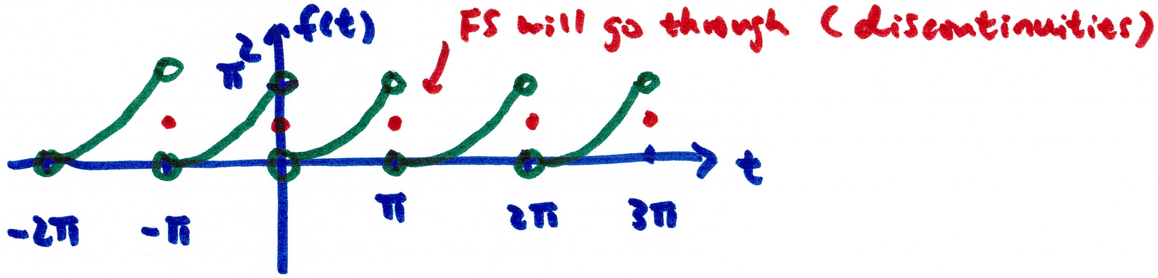
therefore, if $f(t)$ has period $2L$ for $0 < t < 2L$, then its Fourier series is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right]$$

$$a_n = \frac{1}{L}\int_0^{2L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

$$b_n = \frac{1}{L}\int_0^{2L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

example $f(t) = t^2$ for $0 < t < \pi$ with period π



$$a_n = \frac{1}{L}\int_0^{2L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt \qquad L = \frac{\pi}{2} \text{ (half period)}$$

calculate a_0 separately

$$a_0 = \frac{1}{\pi/2}\int_0^{\pi} t^2 \cos\Big(\overset{\swarrow 0}{\frac{n\pi t}{\pi/2}}\Big)dt = \frac{2}{\pi}\int_0^{\pi} t^2\,dt = \frac{2}{3}\pi^2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} t^2\cos(2nt)\,dt$$

$$= \frac{2}{\pi}\left[\frac{(2n^2\pi - 1)\overbrace{\sin(2\pi n)}^{0} + 2\pi n\overbrace{\cos(2\pi n)}^{1}}{4n^3}\right] \quad \text{(from Wolfram)}$$

$$= \frac{1}{n^2}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} t^2 \sin(2nt)\,dt = \cdots = -\frac{\pi}{n}$$

so, the Fourier Series is

$$F(t) = \frac{1}{2}\left(\frac{2}{3}\pi^2\right) + \sum_{n=1}^{\infty}\left[\frac{1}{n^2}\cos(2nt) + -\frac{\pi}{n}\sin(2nt)\right]$$

$$\sim f(t) = t^2$$

converges to

$$F(t) = \frac{1}{3}\pi^2 + (\cos 2t - \pi \sin 2t) + \left(\frac{1}{4}\cos 4t - \frac{\pi}{2}\sin 4t\right)$$

$$+ \left(\frac{1}{9}\cos 6t - \frac{\pi}{3}\sin 6t\right) + \cdots \sim f(t) = t^2$$

be very careful when replacing f(t) with F(t) at discontinuities

for example, $f(t) = t^2 \quad 0 < t < \pi$ period π

is NOT defined at $t = 0$

but $F(0) = \frac{\pi^2}{2}$ $\left(\frac{1}{2}\left[f(t^+) + f(t^-)\right]\right)$

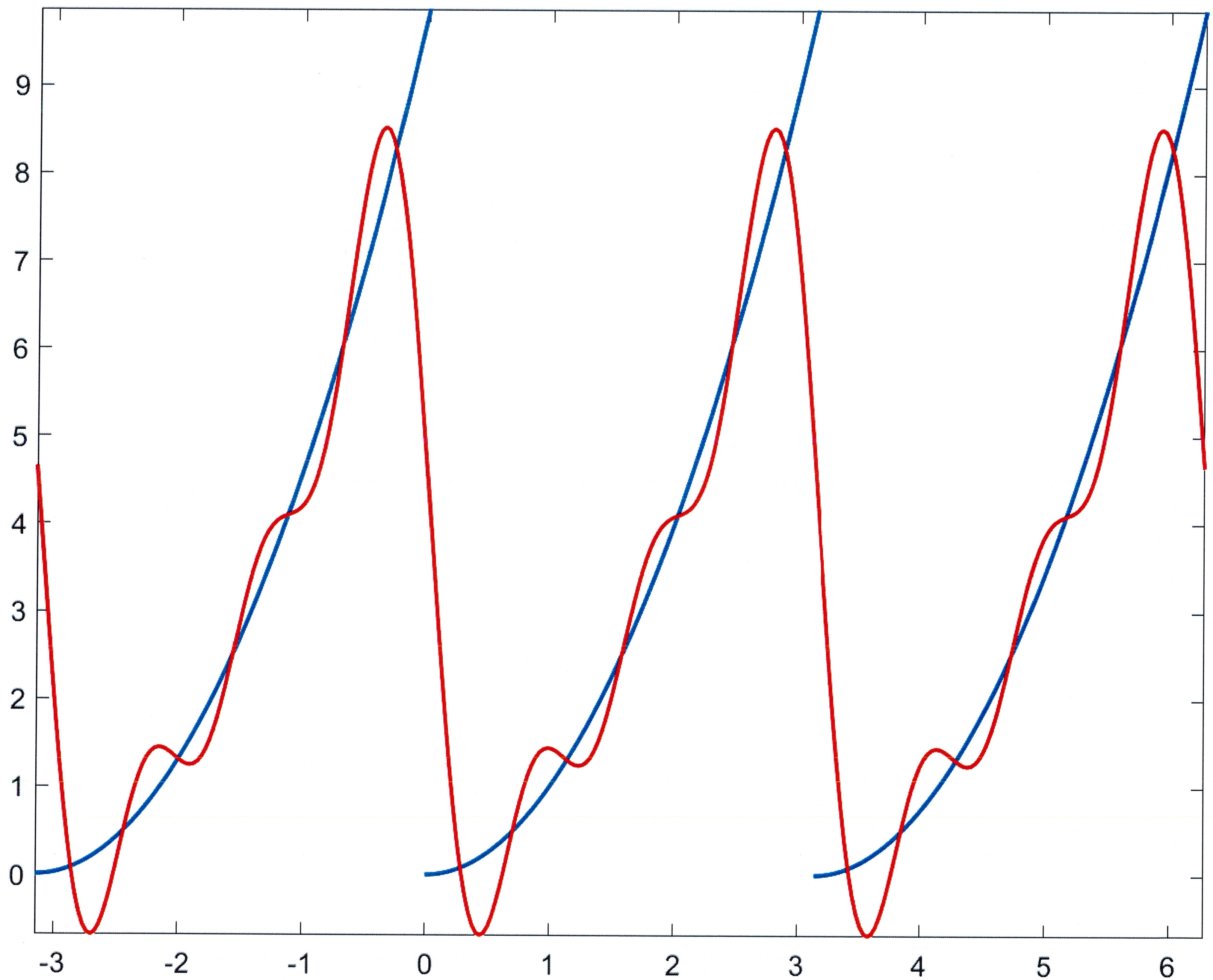
$$F(0) = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{2} \rightarrow \boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}}$$

Basel Problem

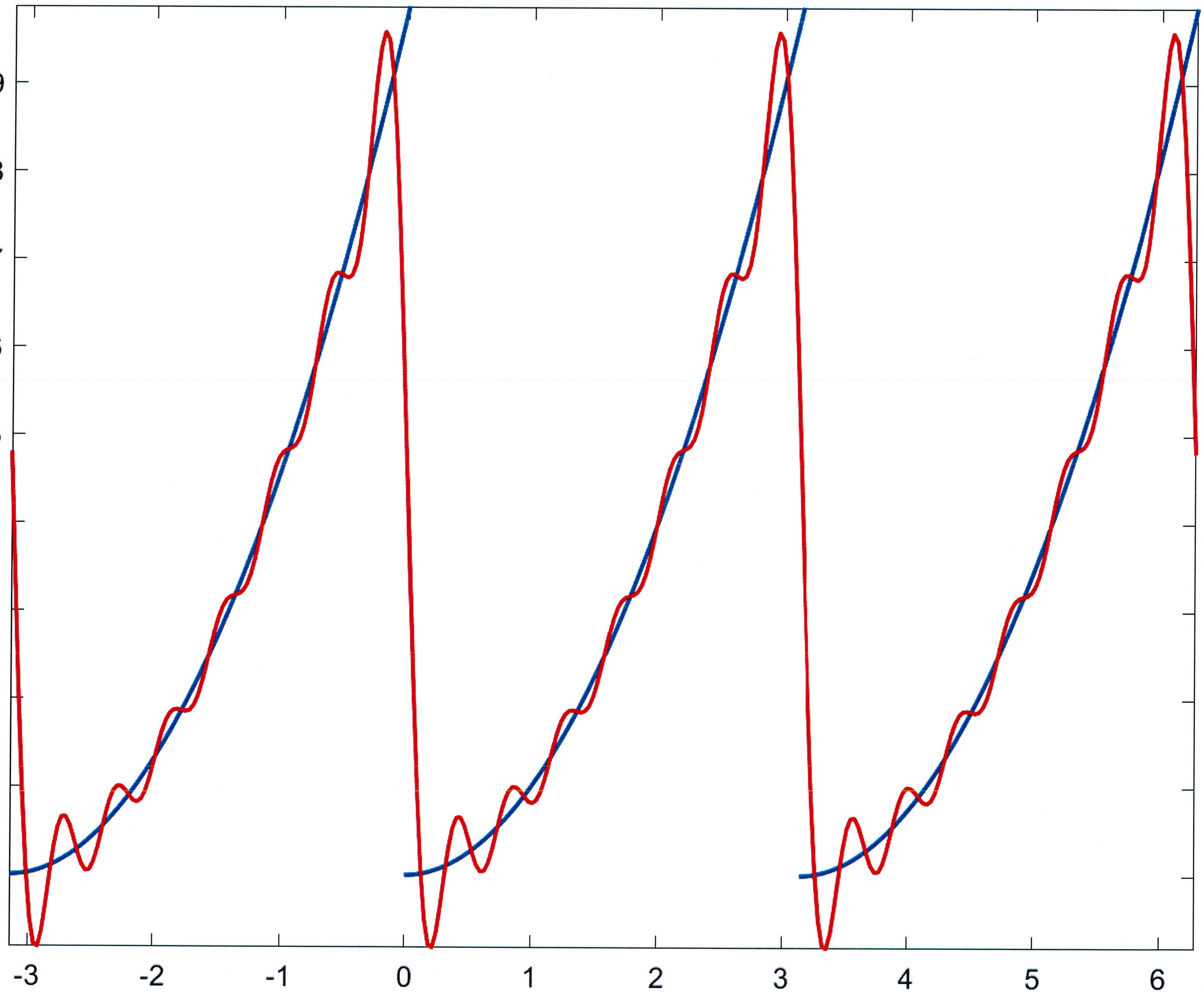
$\downarrow$

Hometown of Euler + Bernoulli

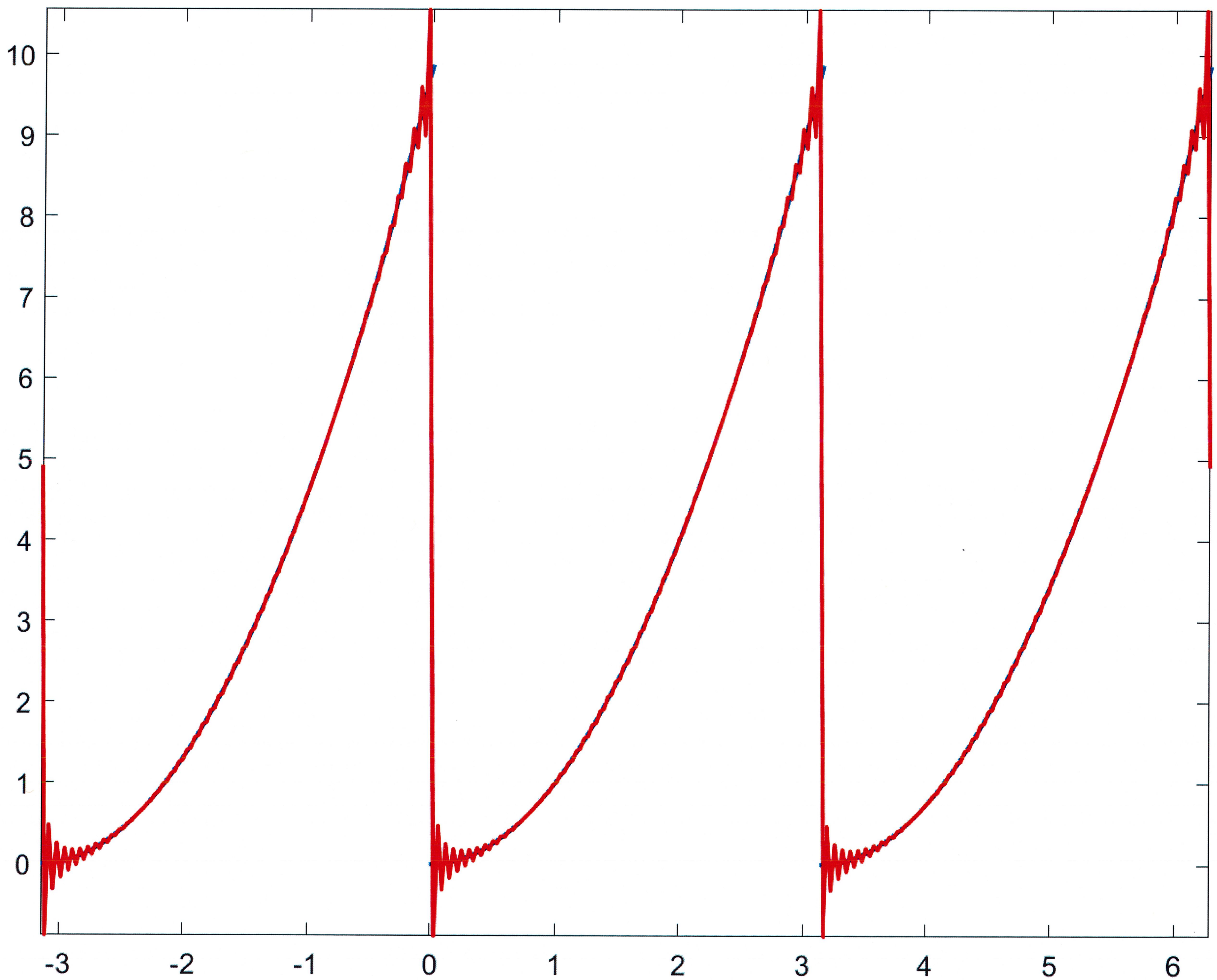
n=3



n=7



n= 50



<u>example</u> $f(t) = \dfrac{\pi^2 - 2\pi t}{8} \quad 0 \le t \le \pi \quad$ period π

$L = \pi/2$

$a_0 = \cdots = 0$

$a_n = \cdots = 0$

$b_n = \cdots = \dfrac{\pi}{4n}$

$$F(t) = \sum_{n=1}^{\infty} \frac{\pi}{4n} \sin(2nt) \sim f(t) = \frac{\pi^2 - 2\pi t}{8}$$

$F(\frac{\pi}{4}) = f(\frac{\pi}{4})$ because $f(t)$ is continuous at $t = \frac{\pi}{4}$

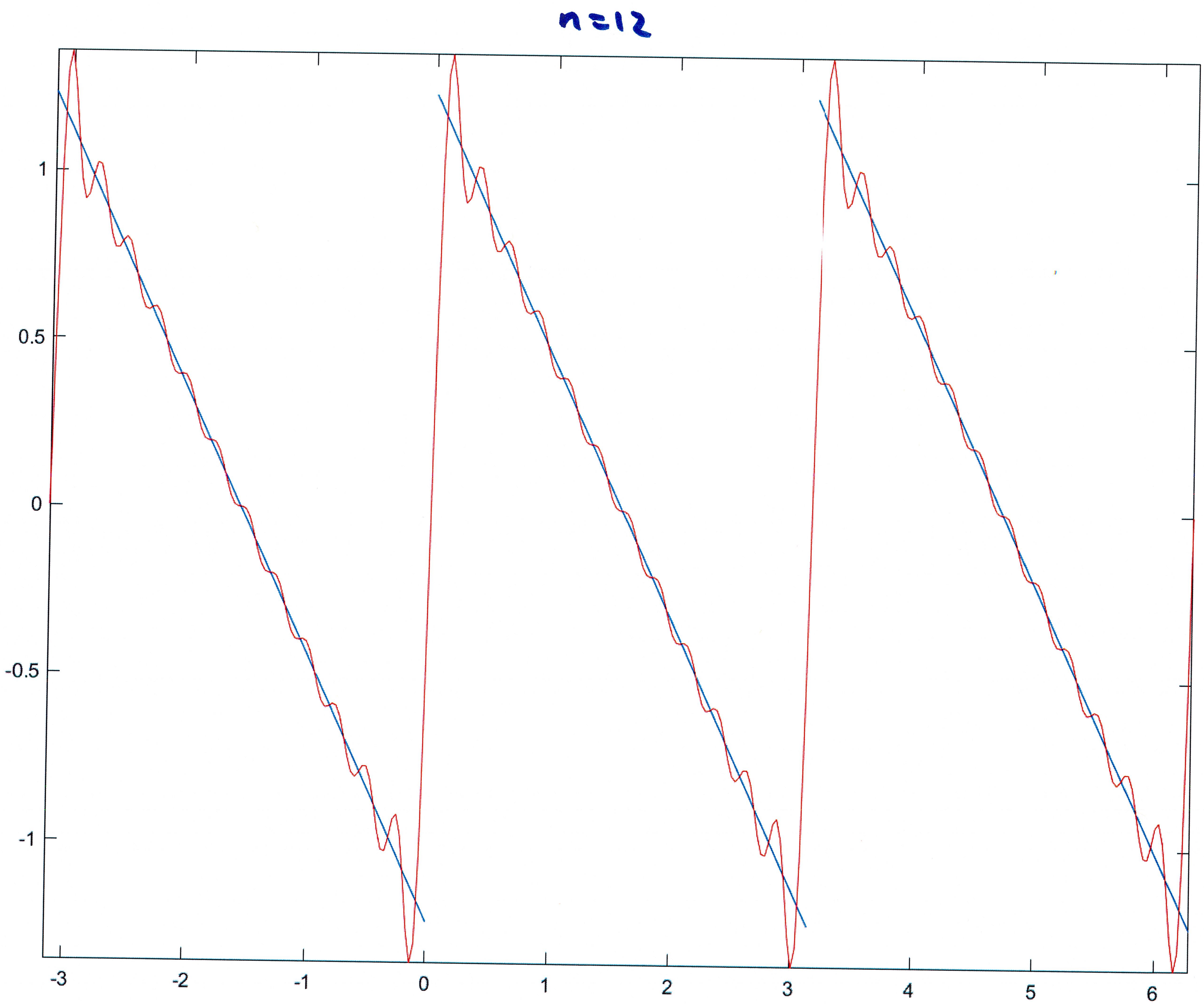
$= \dfrac{\pi^2}{16}$

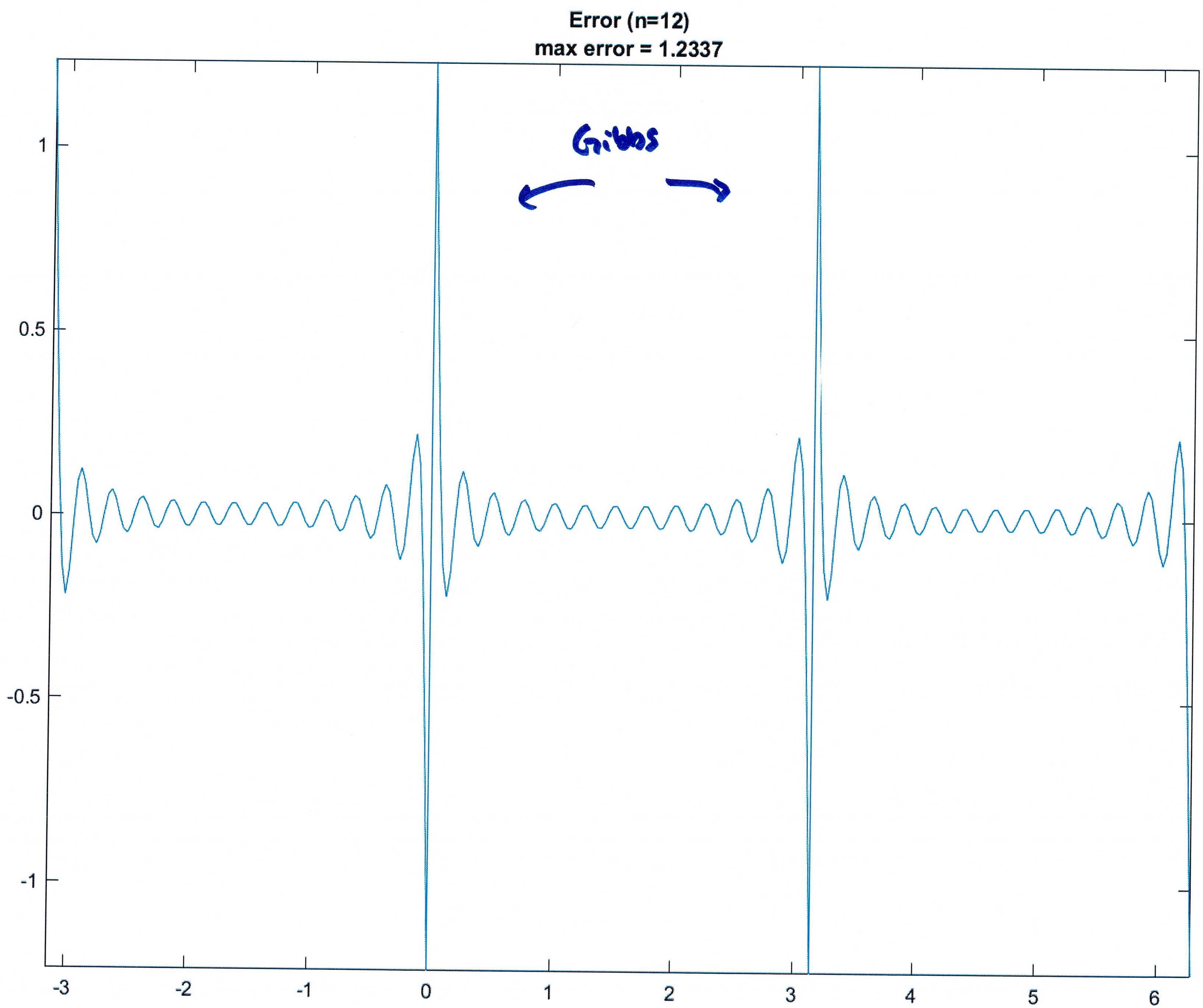
$$F(t) = \frac{\pi}{4}\left(\sin 2t + \frac{1}{2}\sin 4t + \frac{1}{3}\sin 6t + \frac{1}{4}\sin 8t + \frac{1}{5}\sin 10t + \cdots\right)$$

$$F(\tfrac{\pi}{4}) = \frac{\pi}{4}\left(1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} + \cdots\right)$$

$$= \frac{\pi}{4}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\right) = \frac{\pi^2}{16}$$

$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \frac{\pi}{4}$ Leibniz Series


n=12
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
4
5
6


Error (n=12)
max error = 1.2337
Gibbs
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
4
5
6