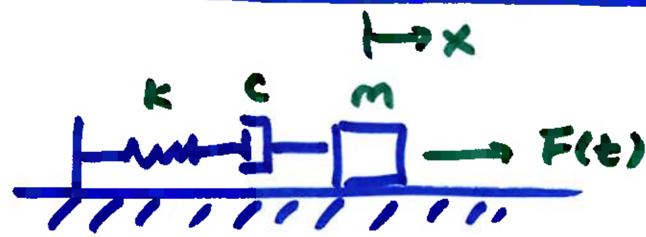
# 9.4 Applications of Fourier Series (part 1)

mass-spring-damper




$x$: displacement from equilibrium

$m$: mass  $k$: spring constant

$c$: damping constant

$F(t)$: external force

$$mx'' + cx' + kx = F(t)$$

let's look at cases w/ $c = 0$, $F(t)$ periodic

$$mx'' + kx = F(t)$$

general solution: $$x(t) = \underbrace{c_1 \cos\sqrt{\frac{k}{m}}\,t + c_2 \sin\sqrt{\frac{k}{m}}\,t}_{\substack{\text{homogeneous part} \\ \text{no } F(t)}} + \underbrace{x_{part}}_{\substack{\text{particular} \\ \text{due } F(t)}}$$

we are interested in $x_{part}$ when $F(t)$ is periodic

**example** $mx'' + kx = F(t)$ $\quad$ $F(t) = \begin{cases} 1 & 0<t<3 \\ -1 & 3<t<6 \end{cases}$

$m = 1, \quad k = 5$

$F(t)$ has period of 6

expand it in a Fourier Sine series (to preserve symmetry with respect to $t=0$)

Sine series: $a_n = 0$

$$b_n = \frac{2}{L}\int_0^L F(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

$L = 3$ (half period)

$$b_n = \frac{2}{3}\int_0^3 1\cdot\sin\left(\frac{n\pi t}{3}\right)dt = \cdots = \frac{2}{n\pi}\left[1-(-1)^n\right]$$

so, $$F(t) = \sum_{n=1}^{\infty}\frac{2}{n\pi}\left[1-(-1)^n\right]\sin\left(\frac{n\pi t}{3}\right)$$

$$= \underbrace{\frac{4}{\pi}\sin\left(\frac{\pi t}{3}\right) + \frac{4}{3\pi}\sin\left(\frac{3\pi t}{3}\right) + \frac{4}{5\pi}\sin\left(\frac{5\pi t}{3}\right) + \cdots}$$

$$x'' + 5x = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right] \sin\left(\frac{n\pi t}{3}\right)$$

$$x = C_1 \cos\sqrt{5}t + C_2 \sin\sqrt{5}t + x_p$$

particular solution from $F(t)$

expand $x_p$ as Fourier series

$$x_p = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi t}{3}\right)$$

$$x_p' = \sum_{n=1}^{\infty} B_n \cdot \frac{n\pi}{3} \cos\left(\frac{n\pi t}{3}\right)$$

$$x_p'' = \sum_{n=1}^{\infty} -\frac{n^2\pi^2}{3^2} B_n \sin\left(\frac{n\pi t}{3}\right)$$

$x'' + 5x = F(t)$

$x_p$ satisfies the ODE

$x_p'' + 5x_p = F(t)$

Since $F(t)$ is a sine series all the cosine terms will end up being zero in $x_p$

$$\sum_{n=1}^{\infty} -\frac{n^2\pi^2}{3^2} B_n \sin\left(\frac{n\pi t}{3}\right) + \sum_{n=1}^{\infty} 5B_n \sin\left(\frac{n\pi t}{3}\right) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right] \sin\left(\frac{n\pi t}{3}\right)$$

So, $B_n = \dfrac{2\left[1-(-1)^n\right]}{n\pi\left(5-\frac{n^2\pi^2}{3^2}\right)}$ and $x_p = \displaystyle\sum_{n=1}^{\infty} \frac{2\left[1-(-1)^n\right]}{n\pi\left(5-\frac{n^2\pi^2}{3^2}\right)} \sin\left(\frac{n\pi t}{3}\right)$ steady periodic solution

$$= 0.3262 \sin\left(\frac{\pi t}{3}\right) - 0.0872 \sin\left(\frac{3\pi t}{3}\right) + \cdots$$

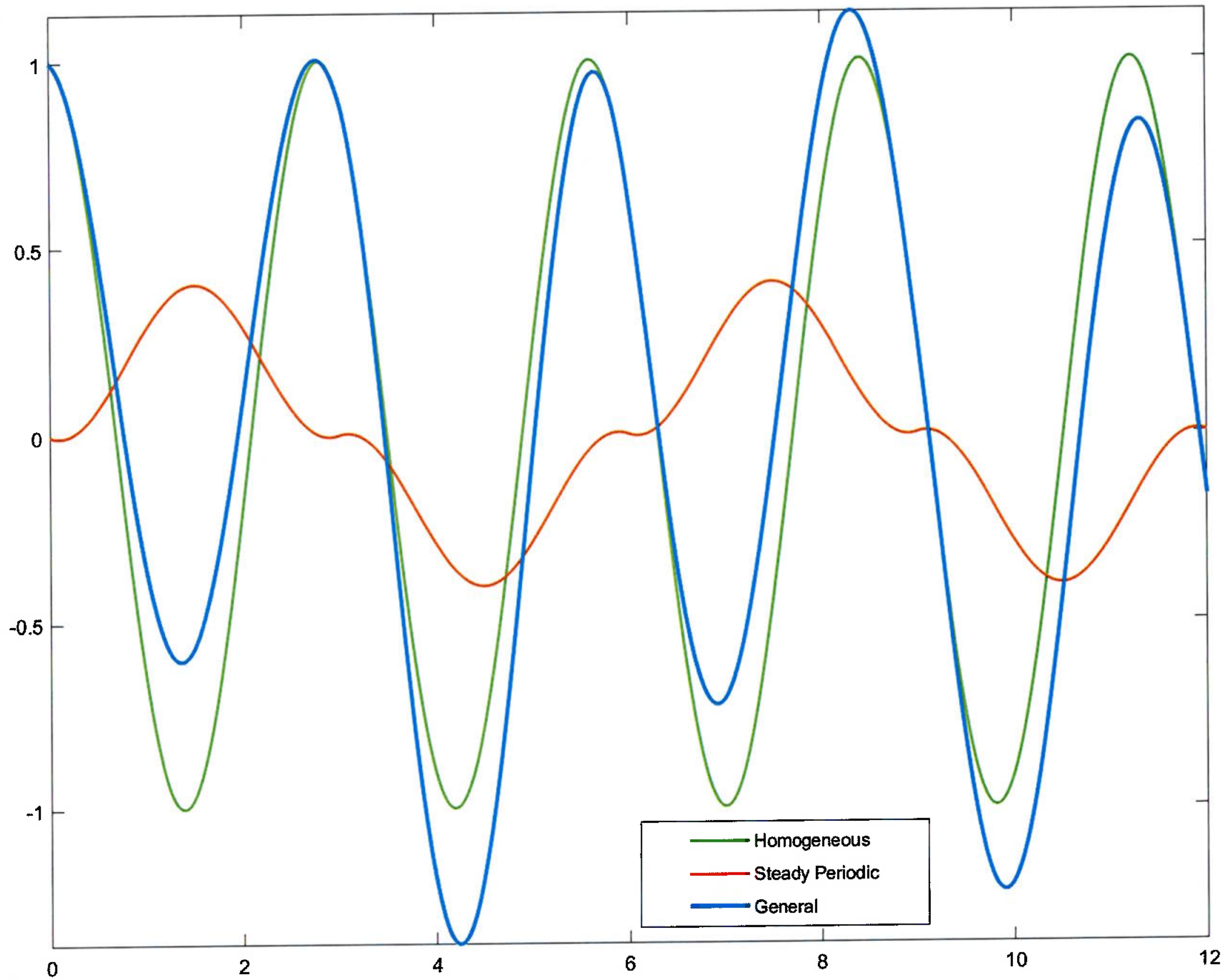
general solution: $x(t) = \underbrace{C_1 \cos\sqrt{5}t + C_2 \sin\sqrt{5}t}_{\text{period } \frac{2\pi}{\sqrt{5}}} + \underbrace{\sum_{n=1}^{\infty} \frac{2[1-(-1)^n]}{n\pi\left(5 - \frac{n^2\pi^2}{3^2}\right)} \sin\left(\frac{n\pi t}{3}\right)}_{\text{period } \frac{6}{n} = \frac{2\pi}{n\pi/3}}$

ratio is NOT rational

the total is NOT periodic

general solution is NOT periodic

graph of each w/ $x(0) = 1$, $x'(0) = 0$

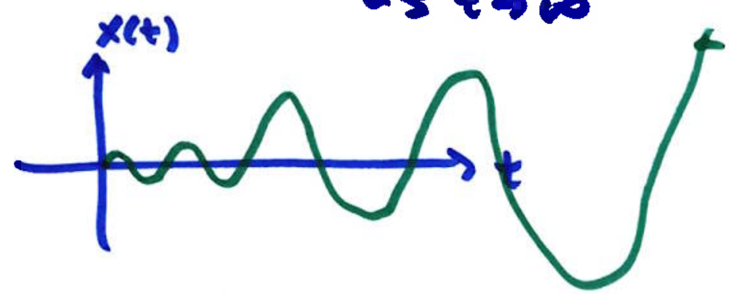

Homogeneous
Steady Periodic
General
1
0.5
0
-0.5
-1
0
2
4
6
8
10
12

in the example, $F(t)$ never contains a $\sin(\sqrt{5}t)$ which duplicates the homogeneous part

what happens if $F(t)$ matches the frequency?

e.g. $x'' + 5x = \sin(\sqrt{5}t) \rightarrow$ resonance

$$x(t) = c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t) - \underbrace{\frac{1}{2\sqrt{5}} t \cos(\sqrt{5}t)}_{\text{resonance magnitude} \to \infty \text{ as } t \to \infty}$$




sometimes, part of $F(t)$ may match or nearly match the frequency of homogeneous part __example__ $x'' + 5x = F(t)$ $\quad F(t) = \begin{cases} 1 & 0 < t < 10 \\ -1 & 10 < t < 20 \end{cases}$ period 20

following the same steps, we get

$$F(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right] \sin\left(\frac{n\pi t}{10}\right)$$

and $$x_p = \sum_{n=1}^{\infty} \frac{2\left[1-(-1)^n\right]}{n\pi\left(5-\frac{n^2\pi^2}{10^2}\right)} \sin\left(\frac{n\pi t}{10}\right)$$

$$= 0.26\sin\left(\frac{\pi t}{10}\right) + 0.10\sin\left(\frac{3\pi t}{10}\right) + 0.10\sin\left(\frac{5\pi t}{10}\right) + \underline{1.11}\sin\left(\frac{7\pi t}{10}\right) - 0.05\sin\left(\frac{9\pi t}{10}\right) + \cdots$$

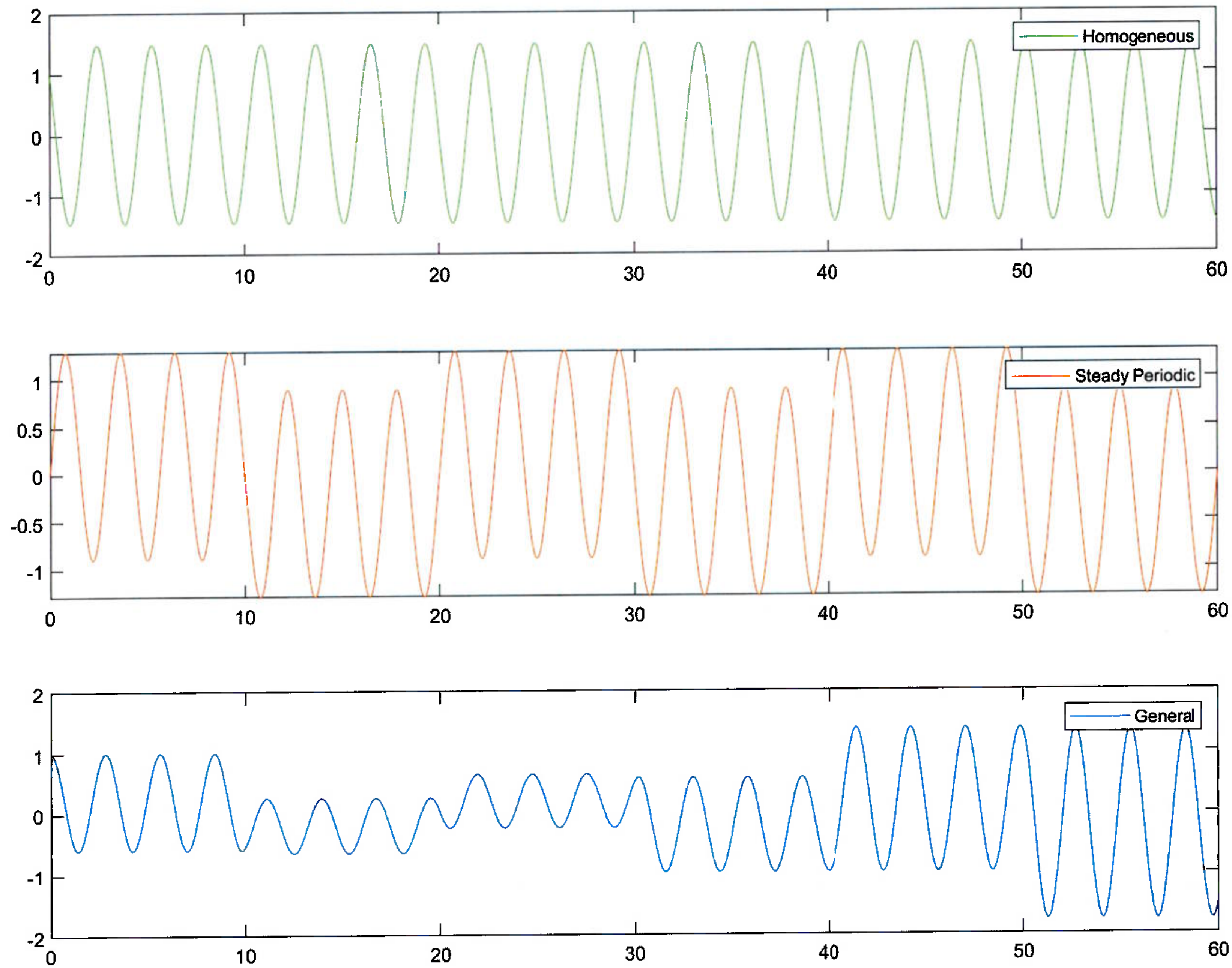
huge!

when $n=7$, $\sin\left(\frac{n\pi t}{10}\right) = \sin\left(\frac{7\pi}{10}t\right) \approx \sin(2.199t)$

$\sin(\sqrt{5}t) \approx \sin(2.236t)$

in homogeneous

this is called __near__ resonance (as opposed to __pure__ resonance)


Homogeneous
Steady Periodic
General
0
10
20
30
40
50
60
2
1
0
-1
-2
0.5
-0.5