9.5 Heat Equation (part 2)

$u_t = ku_{xx}$

Insulated

heat can leave

$x=0$ $\quad$ $x=L$

$u(0,t) = u(L,t) = 0$

$u(x,0) = f(x)$

Solution: $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)$$

now we look at the insulated version

insulated

heat cannot leave

$u_t = ku_{xx}$

BC's: $u_x(0,t) = u_x(L,t) = 0$ → rate $= 0$ → heat cannot travel out the rod

IC: $u(x,0) = f(x)$

apply method of separation again: $u(x,t) = X(x)\,T(t)$

→ $X'(0) = X'(L) = 0$

Same eigenvalues : $\lambda_n = \frac{n^2\pi^2}{L^2}$

eigenfunctions: $X_n = \cos\left(\frac{n\pi x}{L}\right)$

Same T_n: $T_n = e^{-kn^2\pi^2 t/L^2}$

Solutions: $u_n = e^{-kn^2\pi^2 t/L^2} \cos\left(\frac{n\pi x}{L}\right)$ $\quad n = 0, 1, 2, 3, ..$

$u_0 = 1$

general solution:

$$\boxed{u(x,t) = \frac{1}{2} a_0 u_0 + \sum_{n=1}^{\infty} a_n e^{-kn^2\pi^2 t/L^2} \cos\left(\frac{n\pi x}{L}\right)}$$

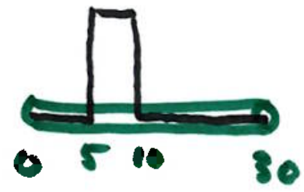
$u(x,0) = f(x)$

$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$ $\quad$ cosine series

$$\boxed{a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx}$$

example Insulated ends

$L=30, \quad K=1$ $\qquad f(x)=\begin{cases} 0 & 0<x<5 \\ 25 & 5<x<10 \\ 0 & 10<x<30 \end{cases}$



physical intuition: temperature evens out as $t\to\infty$

temp expected to be $\dfrac{(5)(25)}{30} = \text{avg.} = \dfrac{25}{6}$

put $L=30$, $K=1$, $f(x)$ into solution on last page

$a_0 = \cdots = \dfrac{25}{3} \qquad a_n = \cdots = \dfrac{50}{n\pi}\left[\sin\left(\dfrac{n\pi}{3}\right) - \sin\left(\dfrac{n\pi}{6}\right)\right]$

first few terms:

$$u(x,t) = \frac{25}{6} + 3.5e^{-\pi^2 t/900}\cos\left(\frac{\overset{1}{n}\pi x}{30}\right) + -3.2\underbrace{e^{-9\pi^2 t/900}}_{\text{large exponent}}\cos\left(\frac{3\pi x}{30}\right)$$

$$-4.1\underbrace{e^{-16\pi^2 t/900}}_{\text{large exponent}}\cos\left(\frac{4\pi x}{30}\right) + \cdots$$

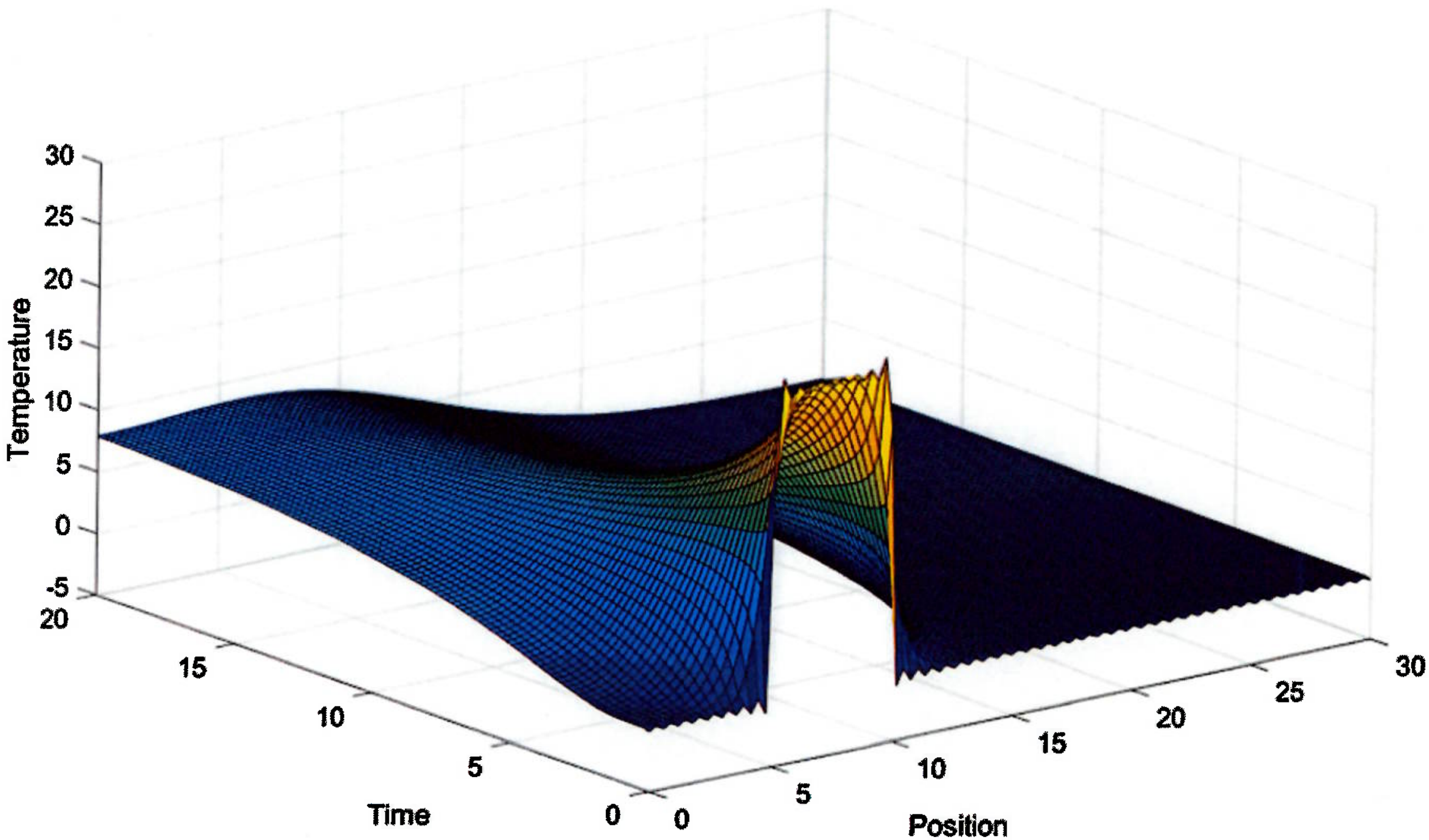
negative exponential $\to$ fast convergence

as $t\to\infty$ $\quad u\to\dfrac{25}{6}$

temp $= \dfrac{25}{6}$

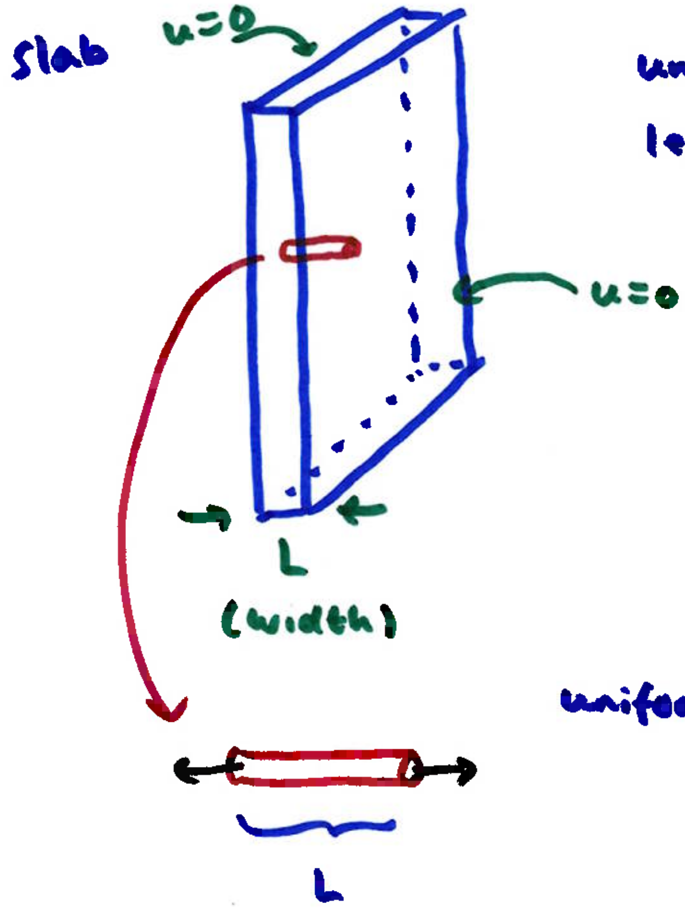
steady-state solution

transient solution : before $t\to\infty$


Temperature
30
25
20
15
10
5
0
-5
20
15
10
5
0
Time
0
5
10
15
20
25
30
Position

back to the non-insulated solution

this can be applied to something like this:



uniformly heated

left and right faces are kept at 0 at all t

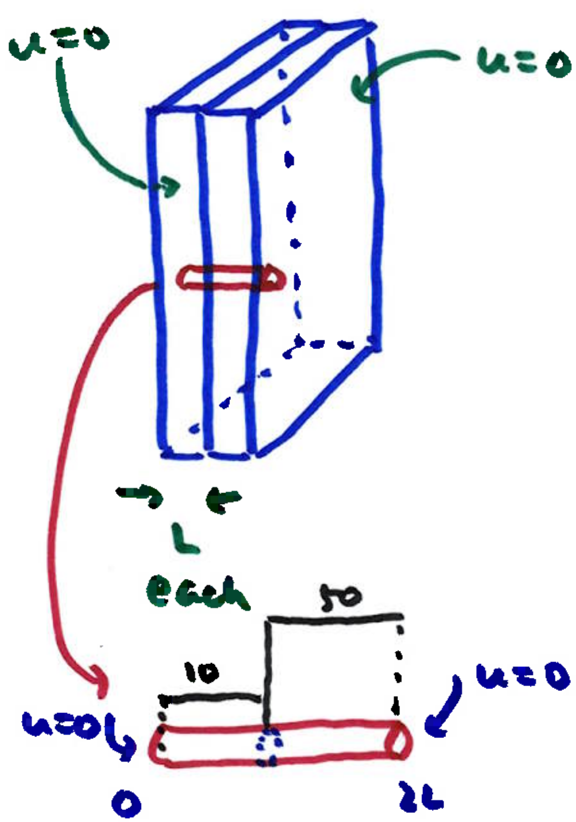
uniformly heated → no temp gradient laterally

no lateral heat flow

→ as if laterally insulated

↑

solve w/ basic heat equation solution we derived two such slabs ~~stake~~ stacked together



uniformly heated (each slab)

two outer faces held at temp = 0

each slab at potentially different temps

for example, left slab at 10

right slab at 50

→ same as rod Length 2L

$u(0,t) = u(2L,t) = 0$

$$f(x) = \begin{cases} 10 & 0<x<L \\ 50 & L<x<2L \end{cases}$$

can reuse solution derived

if not uniformly heated → 2D or 3D heat equation

$$u_t = k(u_{xx} + u_{yy})$$

__example__ : slab 4 cm thick made of copper ($k = 1.15$ cm^2/s)

Both faces kept at $0°C$

Initially entire interior heated to $100°C$

Find temp at center 3 seconds later

How long before center cools to $5°C$?

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$L = 4$ $\quad f(x) = u(x,0) = 100 \quad u(0,t) = u(4,t) = 0$

plug in, find b_n

$$u(x,t) = \sum_{n=1}^{\infty} \frac{200}{n\pi}\left[1-(-1)^n\right] e^{-1.15n^2\pi^2 t/16} \sin\left(\frac{n\pi x}{16}\right)$$

negative exponential fast conv.

center temp 3 sec later

$u(2,3) \approx 15.16°C$

3 terms enough (3 non-zero)

How long before center cools to $5°C$?

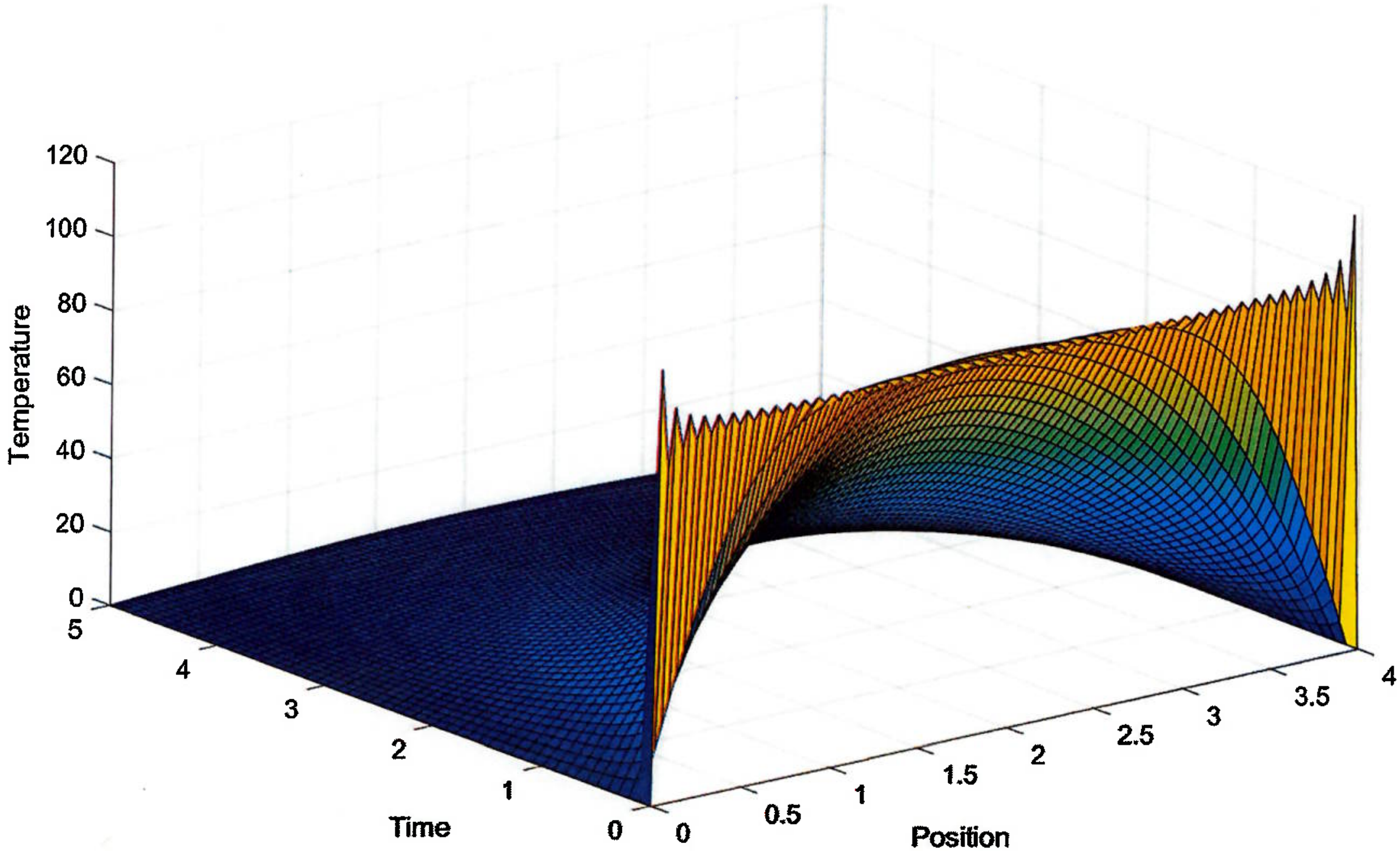
Even longer time, fast converging series $\rightarrow$ 1 non-zero term

$$u(x,t) = \frac{400}{\pi} e^{-1.15\pi^2 t/16} \sin\left(\frac{\pi x}{4}\right)$$

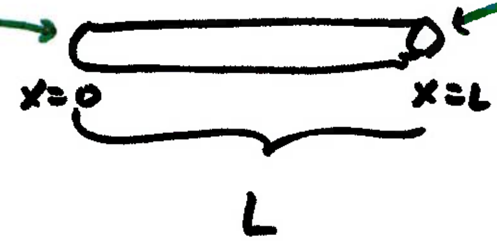
Center: $x = 2$

$$u(2,t) = \frac{400}{\pi} e^{-1.15\pi t^2/16} \underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} = 5$$

Solve for t: $t \approx 4.6$ seconds


Temperature
120
100
80
60
40
20
0
5
4
3
2
1
0
Time
0
0.5
1
1.5
2
2.5
3
3.5
4
Position

now we will let the boundary conditions be nonhomogeneous

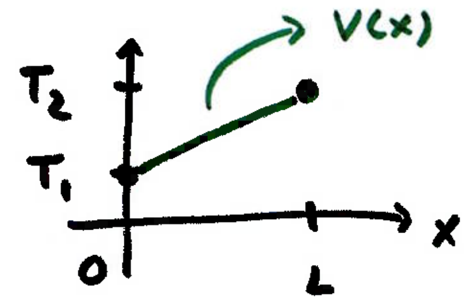
$u(0,t)=T_1$ 

$u(L,t)=T_2$

now T_1, T_2 can be nonzero

$$u_t = k u_{xx}$$

let $V(x) = T_1 + \frac{T_2 - T_1}{L} x$



let $w(x,t) = u(x,t) - V(x)$

note $\boxed{\begin{aligned} w(0,t) &= u(0,t) - V(0) = 0 \\ w(L,t) &= u(L,t) - V(L) = 0 \end{aligned}}$

note $w(x,t)$ has homogeneous BC's

$$u_t = \frac{\partial}{\partial t}(u) = \frac{\partial}{\partial t}(w+v) = w_t + 0 \longleftrightarrow u_t = w_t$$

$$u_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(u)\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(w+v)\right) = \frac{\partial}{\partial x}\left(w_x + \frac{T_2 - T_1}{L}\right)$$

$= W_{xx} + 0 \rightarrow U_{xx} = W_{xx}$

so, $U_t = kU_{xx} \Longrightarrow$

$$\boxed{\begin{aligned} W_t &= k W_{xx} \\ w(0,t) &= 0 \\ w(L,t) &= 0 \\ w(x,0) &= u(x,0) - V(x) \\ &= f(x) - V(x) \end{aligned}}$$

same as before, except the final BC

solution: $w(x,t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$

or, since $u(x,t) = V(x) + w(x,t)$

$$u(x,t) = V(x) + \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

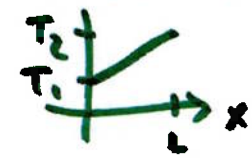
$$\boxed{u(x,t) = \left(T_1 + \frac{T_2 - T_1}{L}x\right) + \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)}$$

at $t=0$, $u(x,0) = f(x)$

$$\hookrightarrow \left[f(x) - \left(T_1 + \frac{T_2 - T_1}{L}x\right)\right] = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\boxed{b_n = \frac{2}{L}\int_0^L \left[f(x) - \left(T_1 + \frac{T_2 - T_1}{L}x\right)\right] \sin\left(\frac{n\pi x}{L}\right) dx}$$

solution as $t \to \infty$ $\quad u(x,t) \to \left(T_1 + \frac{T_2 - T_1}{L}x\right)$ steady-state solution is



steady state solution could have been found from heat eq. easily

$u_t = k u_{xx}$ $\quad$ when time doesn't matter anymore $\to u_t = 0$

$k u_{xx} = 0$

$u_{xx} = 0$

$u_x = C_1$

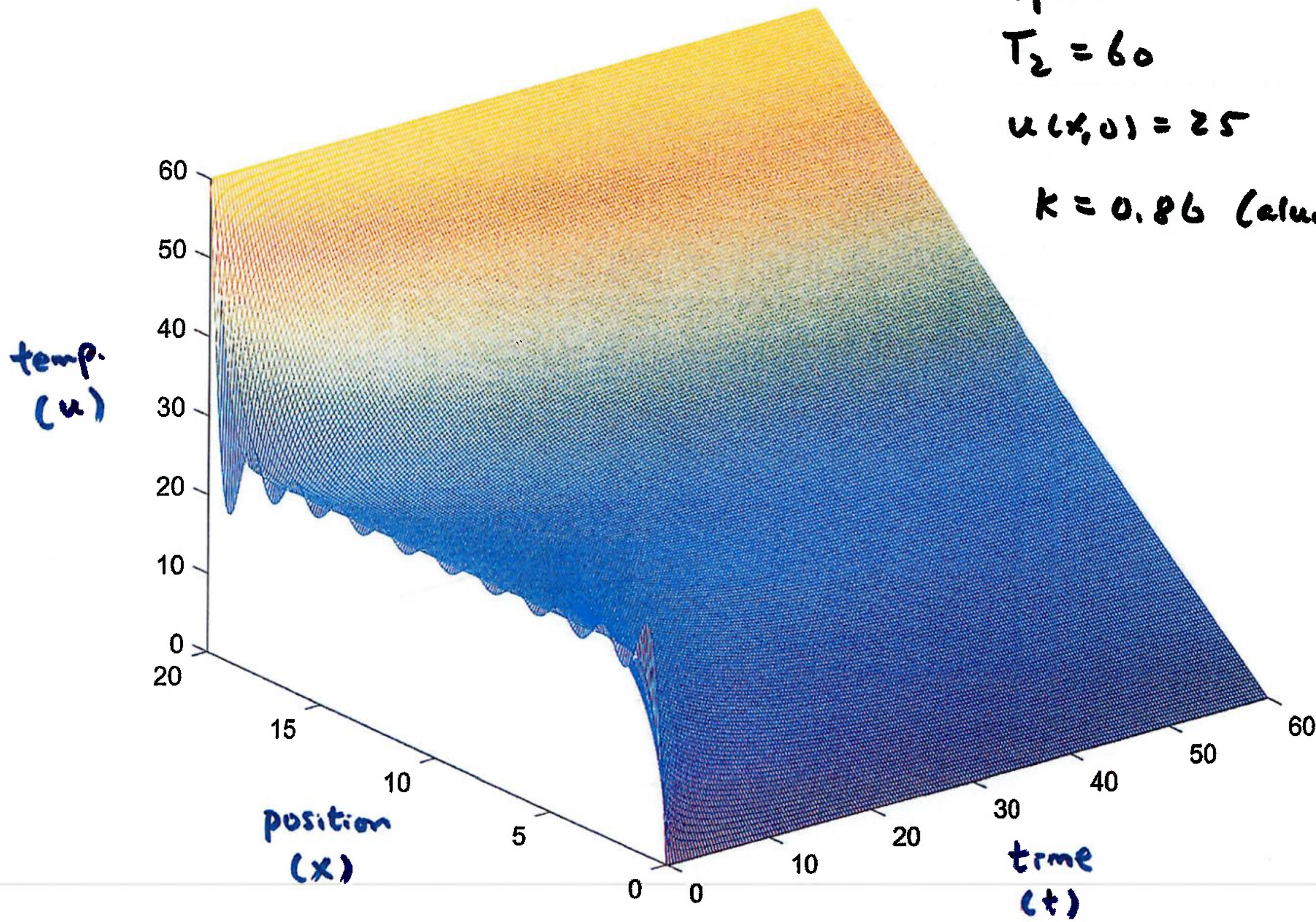
$u = C_1 x + C_2 \quad \to$ same as $\left(T_1 + \frac{T_2 - T_1}{L}x\right)$

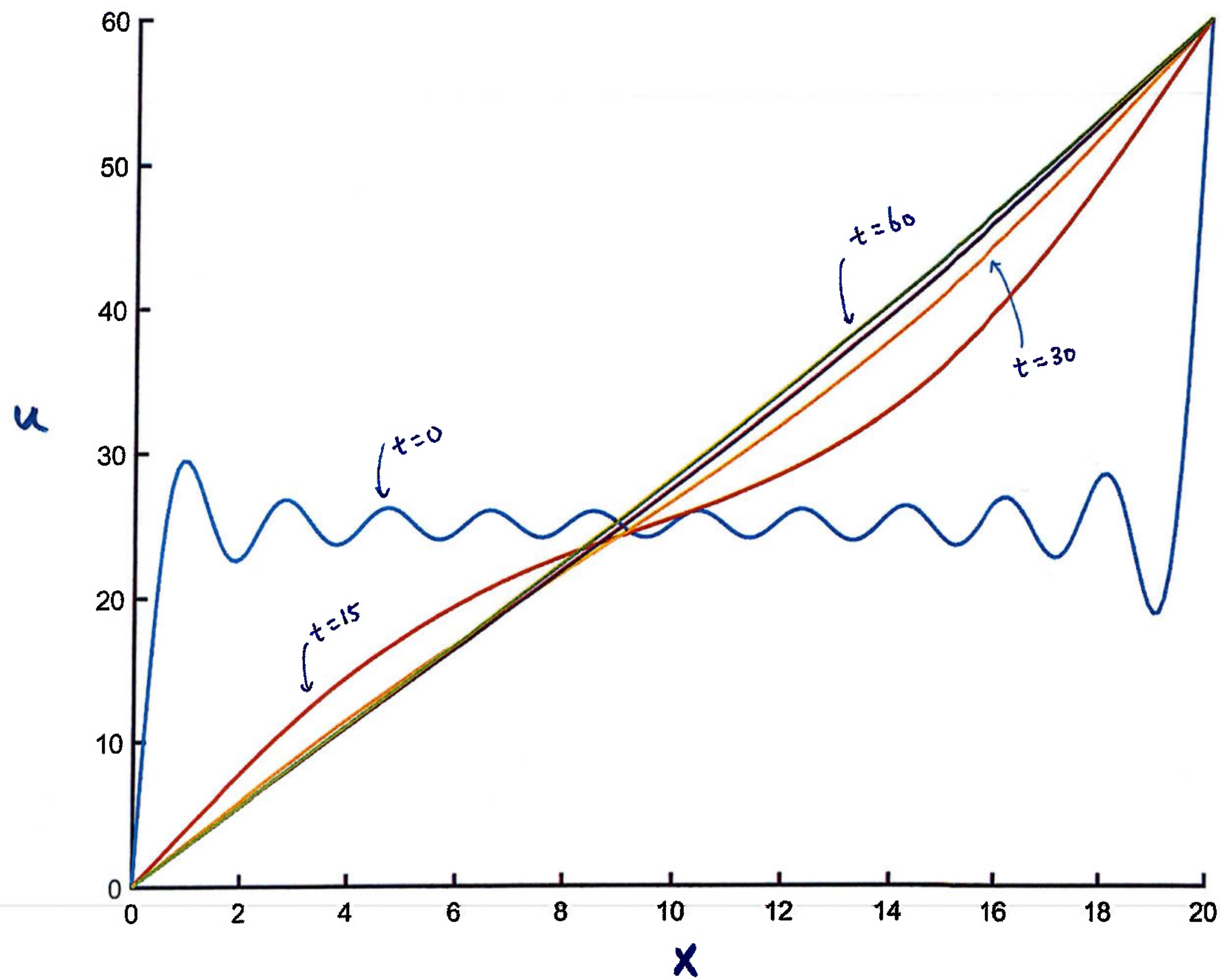
$T_1 = 0$

$T_2 = 60$

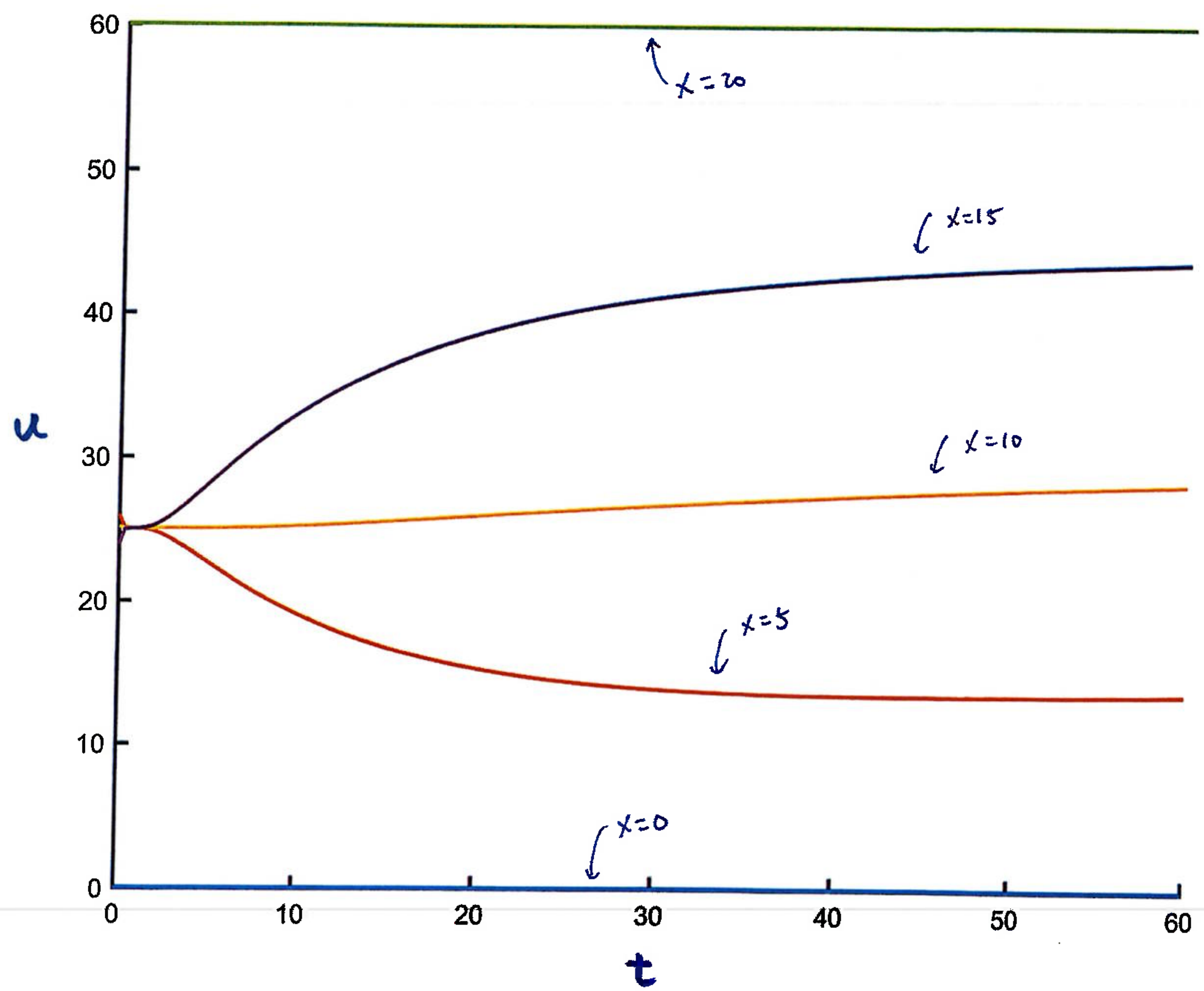
$u(x,0) = 25$

$k = 0.86$ (aluminum)





u
x
0
10
20
30
40
50
60
0
2
4
6
8
10
12
14
16
18
20
t=0
t=15
t=30
t=60



u
t
0
10
20
30
40
50
60
x=20
x=15
x=10
x=5
x=0