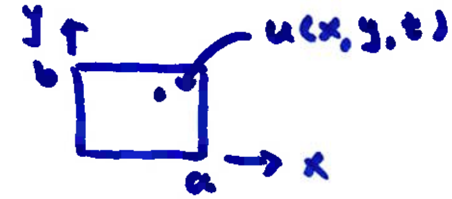
9.7 Steady-State Temp. and Laplace's Equation

1-D Heat Eq: $u_t = k u_{xx}$ $\quad 0<x<L$

2-D Heat Eq: $u_t = k(u_{xx} + u_{yy})$ $\quad 0<x<a$, $0<y<b$



3-D Heat Eq: $u_t = k(u_{xx} + u_{yy} + u_{zz})$

the right side, regardless of dimensions, is simply $\nabla^2 u$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \cdots$$

if $u(x,t)$ (1-D)

$$\nabla^2 u = \frac{\partial^2}{\partial x^2}(u) = u_{xx}$$

if $u(x,y,t)$ (2-D)

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy}$$

wave Eq: $\left.\begin{aligned} &u_{tt} = a^2 u_{xx} \quad (1\text{-D}) \\ &u_{tt} = a^2(u_{xx} + u_{yy}) \end{aligned}\right\}$ $u_{tt} = a^2 \nabla^2 u$

$\nabla^2 u$ gives the comparison of u at a point compared avg u nearby

1-D Heat $\quad u_t = K\nabla^2 u = K u_{xx}$

$\nabla^2 u < 0$

u here is > avg u nearby

$u_t = K\nabla^2 u$

steady-state → time doesn't affect u → $u_t = 0$

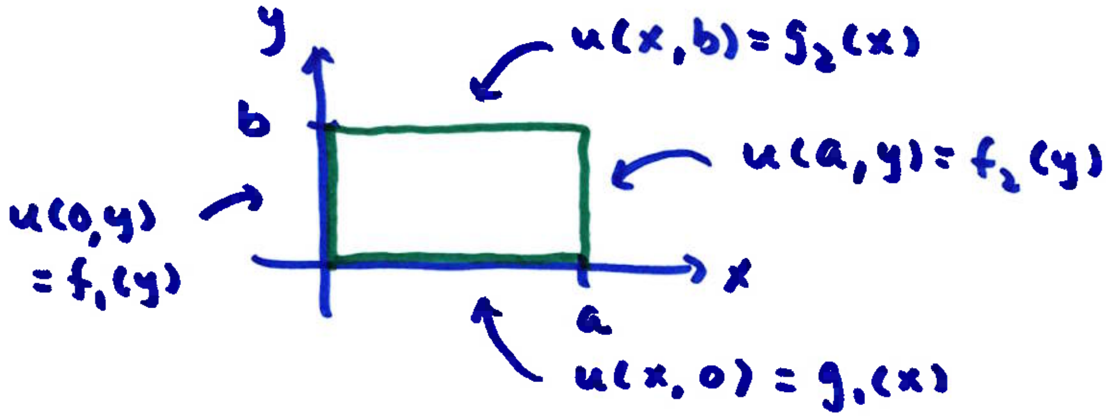
$0 = K\nabla^2 u \rightarrow \boxed{\nabla^2 u = 0}$ <u>Laplace's Equation</u>

<u>2-D Laplace's Equation</u>

$\nabla^2 u = 0 \qquad u(x,y)$

$u_{xx} + u_{yy} = 0 \qquad 0 < x < a, \qquad 0 < y < b$

there are four boundary conditions



if $u=0$ on an edge, that is called homogeneous boundary condition

superposition applies : set 3 edges to be zero, solve. then switch the non-zero edge, solve, combine all results (equivalent to Problem A, B in wave)

as an example, let's solve the case $f_1=0$, $g_1=0$, $g_2=0$, $f_2\neq 0$

$u_{xx}+u_{yy}=0 \quad 0<x<a, \quad 0<y<b$

$$\left.\begin{array}{ll} u(0,y)=0 & \text{left} \\ u(x,0)=0 & \text{bottom} \\ u(x,b)=0 & \text{top} \end{array}\right\} \text{homogeneous}$$

$u(a,y)=f(y)$ right

if all four are 0 trivial solution only (we don't care)

$u(x,y) = X(x)Y(y)$ $\quad u_{xx} = X''Y$ $\quad u_{yy} = XY''$

$u(0,y)=0 \rightarrow X(0)Y(y)=0 \rightarrow X(0)=0$

$$\left.\begin{array}{l} u(x,0)=0 \rightarrow X(x)Y(0)=0 \rightarrow Y(0)=0 \\ u(x,b)=0 \rightarrow X(x)Y(b)=0 \rightarrow Y(b)=0 \end{array}\right\} Y \text{ has complete BC's}$$

$u_{xx} + u_{yy} = 0$

$X''Y + XY'' = 0 \rightarrow \underbrace{\frac{X''}{X} = -\frac{Y''}{Y}}_{\text{solve the one w/ complete BC's first (here, } Y)} = c$

this constant c can be negative or positive depending on BC's

$Y'' + cY = 0 \qquad Y(0) = Y(b) = 0$

only sine/cosine can solve this, so $c > 0$

since $c > 0$, let $\lambda = c \quad (\lambda > 0)$

$Y'' + \lambda Y = 0$ identical to space problem in heat eq.

$\boxed{\lambda_n = \frac{n^2\pi^2}{b^2}}$ $\quad \boxed{Y_n = \sin\left(\frac{n\pi y}{b}\right)}$

"L" for y $(0<y<b)$

return to $\dfrac{X''}{X} = c = \lambda = \dfrac{n^2\pi^2}{b^2}$ (came from solving Y eq.)

$X'' - \dfrac{n^2\pi^2}{b^2} X = 0$ only one BC : $X(0) = 0$

$X = c_1 e^{\frac{n\pi}{b}x} + c_2 e^{-\frac{n\pi}{b}x}$

it is a bit more convenient to write

$X = K_1 \cosh\left(\dfrac{n\pi x}{b}\right) + K_2 \sinh\left(\dfrac{n\pi x}{b}\right)$ $X(0) = 0$

$X(0) = 0 \rightarrow K_1 = 0$, so $\boxed{X_n = \sinh\left(\dfrac{n\pi x}{b}\right)}$

$u = XY$

so for $n = 1, 2, 3, \ldots$ $u_n = X_n Y_n = \sinh\left(\dfrac{n\pi x}{b}\right) \sin\left(\dfrac{n\pi y}{b}\right)$

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right)}$$

one BC left : $u(a,y) = f(y)$ (right edge)

$$u(a,y) = \sum_{n=1}^{\infty} \underbrace{\left[C_n \sinh\left(\frac{n\pi a}{b}\right) \right]}_{\text{constant}} \sin\left(\frac{n\pi y}{b}\right)$$

Sine series w/ $C_n \sinh\left(\frac{n\pi a}{b}\right)$ as ~~const.~~ coeff

Sine series coeff: $\displaystyle \frac{2}{\underset{\uparrow\, b}{L}} \int_0^{L \,\leftarrow\, b} f(y) \sin\left(\frac{n\pi y}{b}\right) dy$

$$C_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b} \int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy$$

$$\boxed{C_n = \frac{2}{b \sinh\left(\frac{n\pi a}{b}\right)} \int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy}$$