6.2 Linear and Almost Linear Systems (part 1)

last time: $\vec{x}' = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix} \vec{x}$

and $\vec{x}' = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

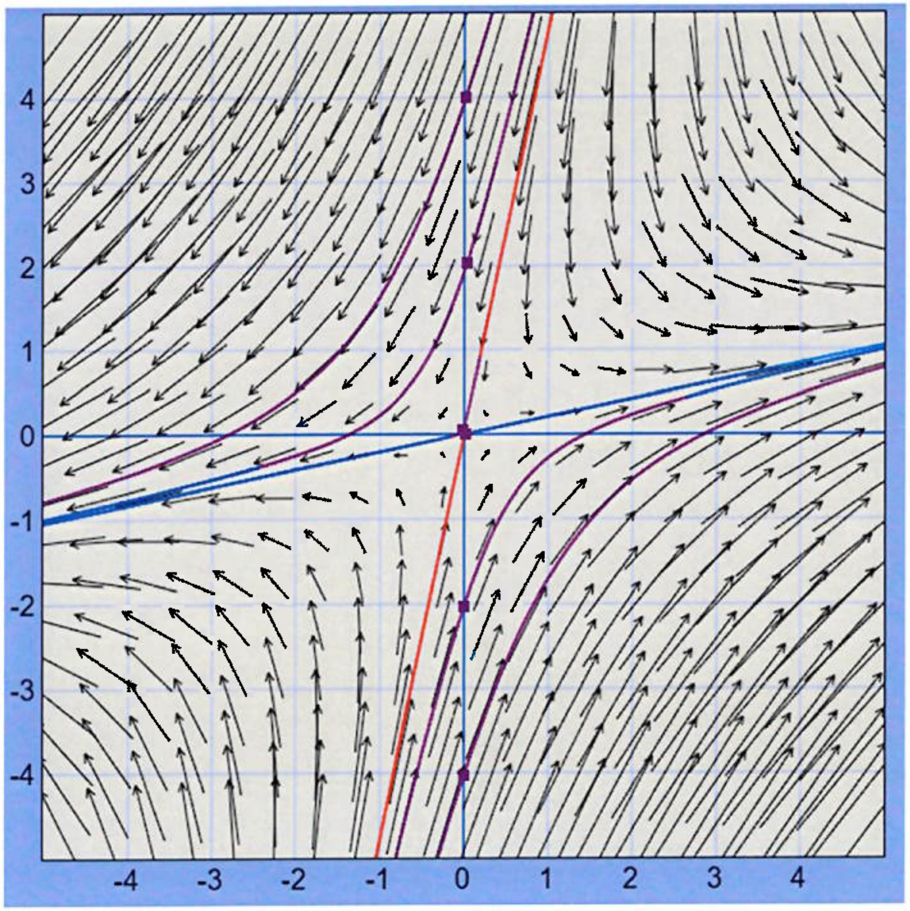
homogeneous one: cp at $(0,0)$

nonhomogeneous one: cp at $(-1,-1)$

their phase diagrams look identical

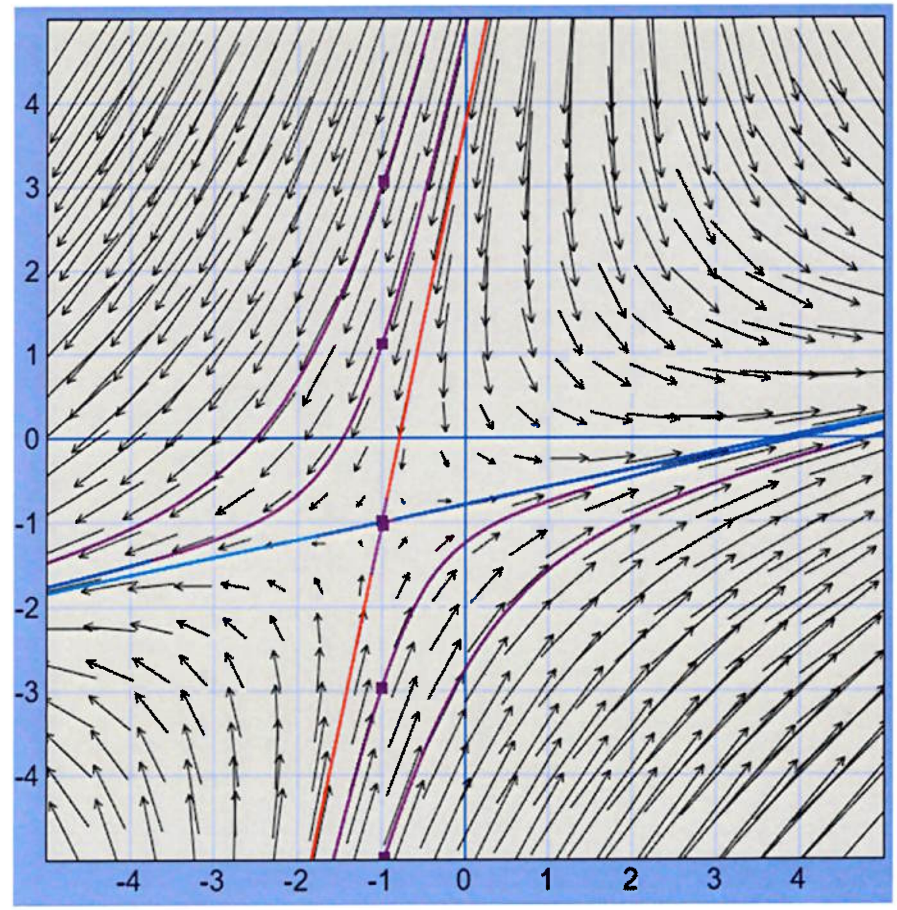
why?

$$\vec{x}' = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$



$$\boldsymbol{x}' = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix} \boldsymbol{x}$$

Critical pt: (0,0)



$$\boldsymbol{x}' = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Critical pt: (-1, -1)

$x' = 2x - y + 1$

$y' = x - 3y - 2$

cp: $(-1, -1)$

define $u = x - (-1)$ $\quad v = y - (-1)$

$u = x + 1$ $\quad v = y + 1$

$x = u - 1$ $\quad y = v - 1$

$x' = u'$ $\quad y' = v'$

sub

$u' = 2(u-1) - (v-1) + 1$

$u' = 2u - v$

$v' = (u-1) - 3(v-1) - 2$

$v' = u - 3v$

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \rightarrow \text{same as } \vec{x}' = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix} \vec{x}$$

that is why the phase diagrams are identical

just moved

$\vec{x}' = A\vec{x} + \vec{u}$ always looks just like $\vec{x}' = A\vec{x}$

constant

example

$$x' = 3x - 8y - 37$$
$$y' = 2x - 7y - 33$$

$$\vec{x}' = \begin{bmatrix} 3 & -8 \\ 2 & -7 \end{bmatrix} \vec{x} + \underbrace{\begin{bmatrix} -37 \\ -33 \end{bmatrix}}_{\text{constant vector}}$$

so phase diagram looks exactly like that of $\vec{x}' = \begin{bmatrix} 3 & -8 \\ 2 & -7 \end{bmatrix} \vec{x}$

but shifted to another cp

$$\frac{dx}{dt} = 5x + 2y + x^2 + y^2$$

$$\frac{dy}{dt} = -4x - 4y - 3xy$$

clearly not linear, but we can <u>linearize</u> about a point (x_0, y_0)

Taylor's formula:

$$f(x,y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) + \cancel{H}OT's$$

higher-order terms

$$f(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

apply to system

$$\frac{dx}{dt} = 5x + 2y + x^2 + y^2 = f(x,y)$$

$$\frac{dy}{dt} = -4x - 4y - 3xy = g(x,y)$$

this system has, among others, $(0,0)$ as a cp

let's linearize about $(0,0)$

$$f(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0)$$

$$= 0 + (5+2\cdot 0)(x) + (2+2\cdot 0)(y) = 5x+2y$$

$$g(x,y) = \cdots = -4x-4y$$

so, near $(0,0)$, the system behaves like

$$\vec{x}\,' = \begin{bmatrix} 5 & 2 \\ -4 & -4 \end{bmatrix} \vec{x}$$

original : $\vec{x}\,' = \begin{bmatrix} 5 & 2 \\ -4 & -4 \end{bmatrix} \vec{x} + \underbrace{\begin{bmatrix} x^2+y^2 \\ -3xy \end{bmatrix}}$

$\to \vec{0}$ as $(x,y) \to (0,0)$ FASTER than $\vec{x}\,' = A\vec{x}$

<u>Almost Linear System</u>

if $(x_0, y_0) \neq (0,0)$ then the nonlinear part may not vanish as above $\to$ must properly linearize so, ignoring the bigger picture, we know near (0,0) system behaves like $\vec{x}' = \underbrace{\begin{bmatrix} 5 & 2 \\ -4 & -4 \end{bmatrix}}_{\lambda = -3,\, 4} \vec{x}$ saddle point

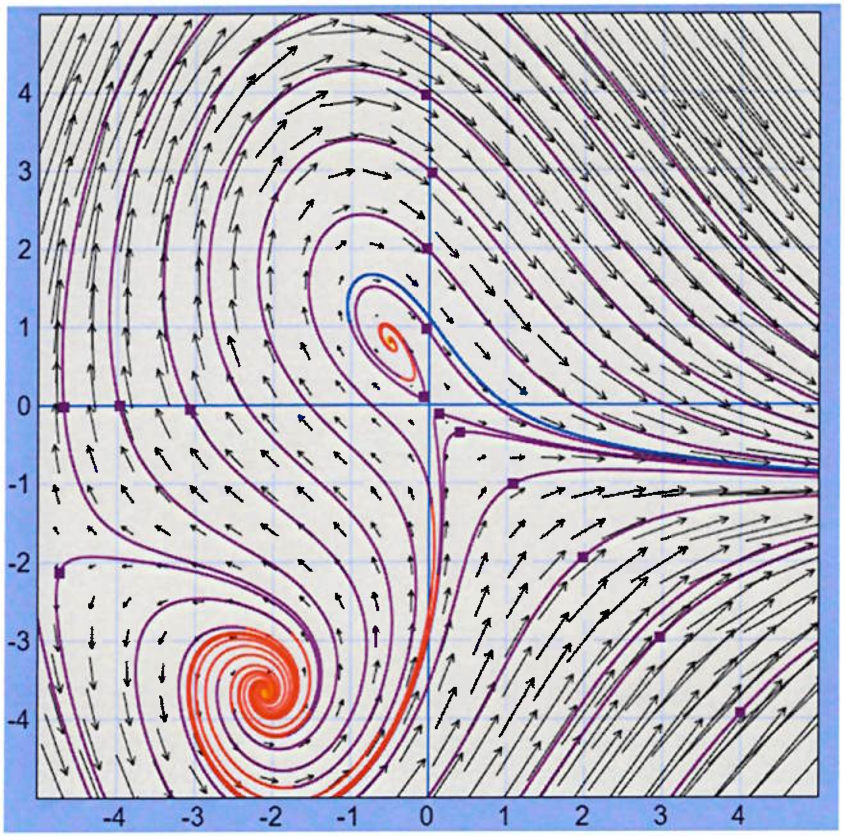
in general, given

$$\frac{dx}{dt} = f(x,y)$$

$$\frac{dy}{dt} = g(x,y)$$

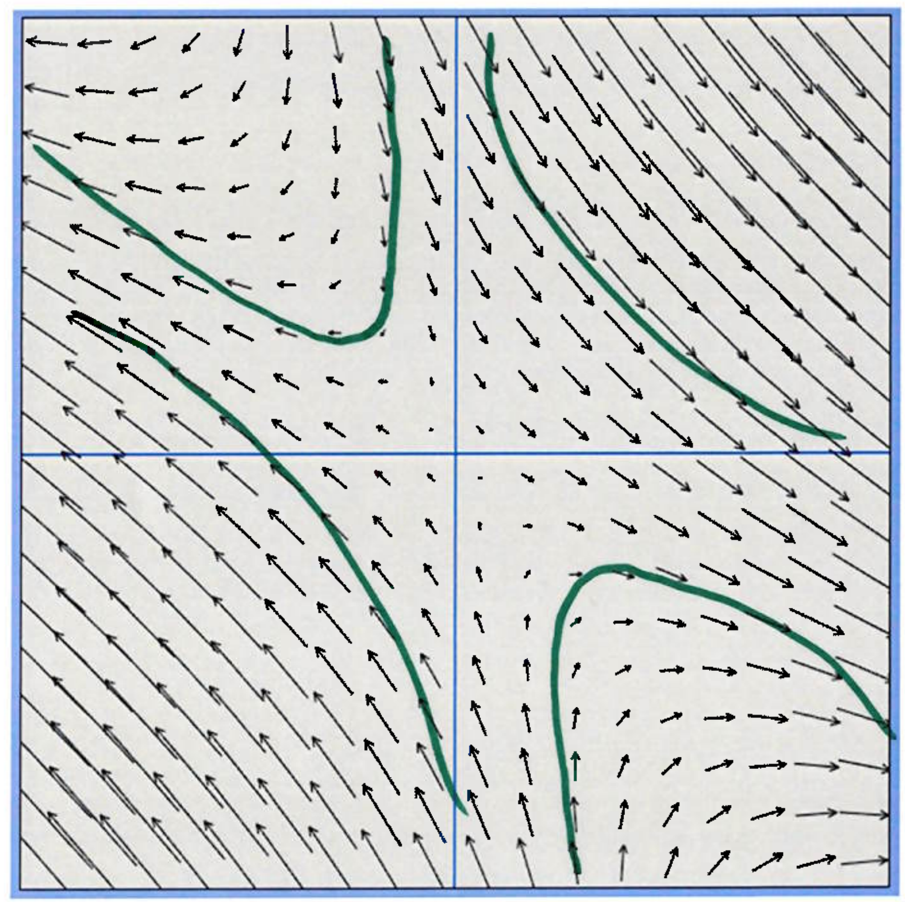
we linearize it about (x_0, y_0) and rewrite it as

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{bmatrix}}_{\text{Jacobian Matrix}} \begin{bmatrix} x \\ y \end{bmatrix}$$



$$x' = 5x + 2y + x^2 + y^2$$
$$y' = -4x - 4y - 3xy$$

Zoomed in near (0,0)



<u>example</u> $\frac{dx}{dt} = -2x - 2y + x^3 = f(x,y)$ $\quad f_x = -2+3x^2 \quad f_y = -2$

$\frac{dy}{dt} = x - 4y + y^4 = g(x,y)$ $\quad g_x = 1 \quad g_y = -4+4y^3$

Jacobian $J = \begin{bmatrix} -2+3x^2 & -2 \\ 1 & -4+4y^3 \end{bmatrix}$

linearized system about $(0,0)$

$$\vec{x}' = \begin{bmatrix} -2 & -2 \\ 1 & -4 \end{bmatrix} \vec{x}$$

near $(0,0)$ ~~then~~ the nonlinear system looks like the linearized system (with some exceptions)

$\lambda = -3 \pm i$

near $(0,0)$ the nonlinear sys should have an asymptotically stable spiral (spiral sink) the nonlinear system behaves like the linearized system or associated linear system near (x_0, y_0) <u>unless</u> :

1) λ's of the linearized system are repeated

2) λ's of the linearized system are purely imaginary

if these happen, the nonlinear system <u>MAY</u> behave differently from the linearized one

we will study this next time