6.2 Linear and Almost Linear Systems (part 2)

last time: the phase diagram of associated linear system resembles that of the nonlinear system near a critical pt **unless** if the associated linear system has:

1) repeated eigenvalues
2) purely imaginary eigenvalues

example

$$\frac{dx}{dt} = -3y + ay(x^2+y^2) = f \qquad a: \text{some constant}$$

$$\frac{dy}{dt} = 3x + ay(x^2+y^2) = g$$

by inspection, one of the cp's is $(0,0)$

linearize about $(0,0)$

$$\vec{x}\,' = \begin{bmatrix} f_x(0,0) & f_y(0,0) \\ g_x(0,0) & g_y(0,0) \end{bmatrix} \vec{x}$$

$f_x = 2axy \qquad f_y = -3+3ay^2+2ax^2y$

$g_x = 3+2axy \qquad g_y = 3ay^2$

$$= \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} \vec{x} \qquad \lambda = \pm 3i$$

so linearized system is suggesting $(0,0)$ is a center (stable)

but the true behavior can be different.

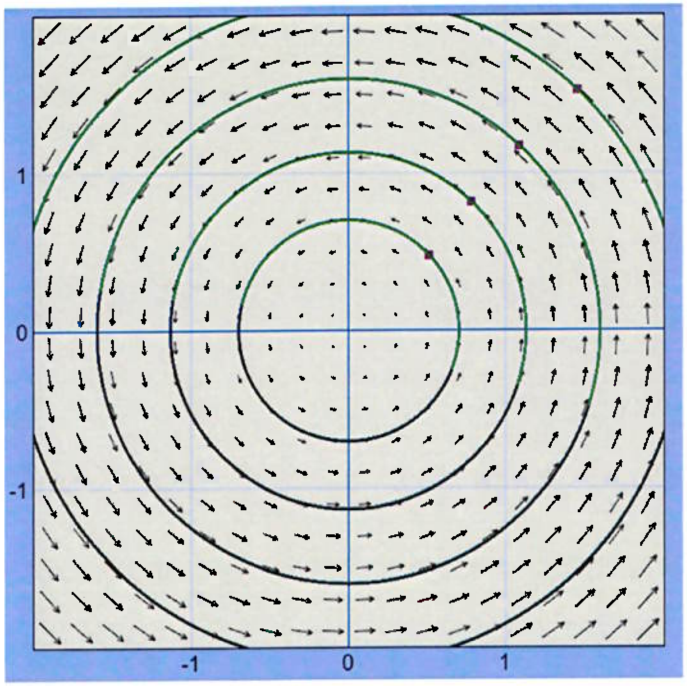
$$\vec{x}\,' = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} \vec{x} + a \begin{bmatrix} y(x^2+y^2) \\ y(x^2+y^2) \end{bmatrix}$$

let's see the phase diagrams of cases when $a = 1$, and $a = -1$

($a = 0 \to$ center)

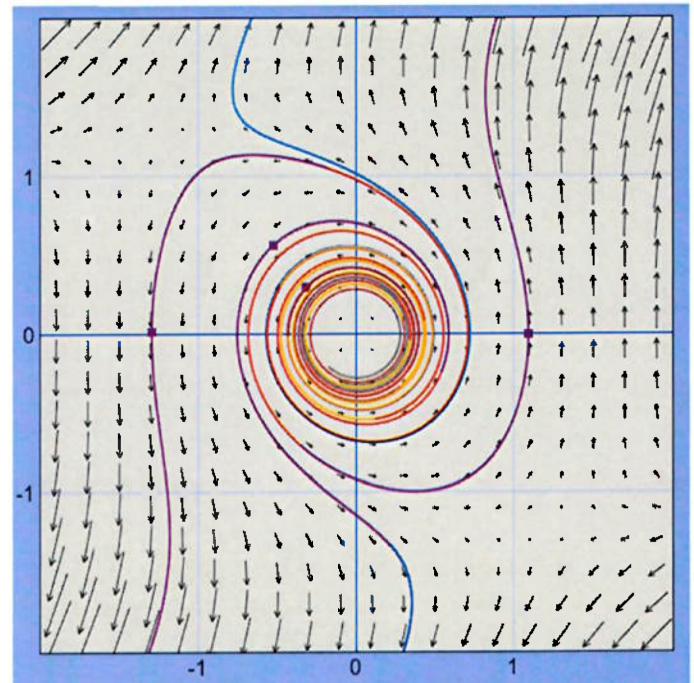
in general, nonlinear systems are hard to predict

$$\left.\begin{array}{l} \text{case here: } \lambda = \text{purely imaginary} \\ \text{similar thing if } \lambda = \text{repeated} \end{array}\right\} \text{"weak" behavior}$$

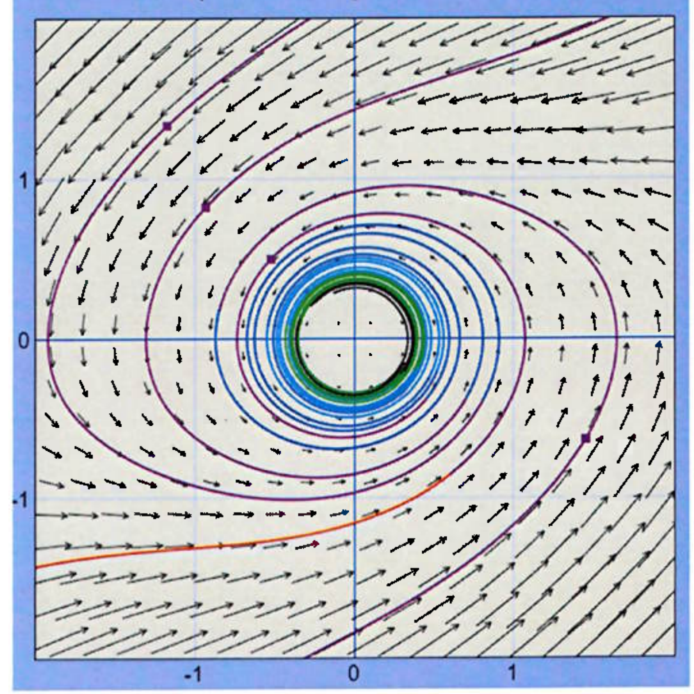


Linearized
(stable center)

Nonlinear a=1
(unstable
spiral point)



Nonlinear a= -1
(asymptotically
stable spiral
point)



$\vec{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x}$ linear system

eigenvalues: $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$ $(a-\lambda)(d-\lambda) - bc = 0$

$\lambda^2 - (a+d)\lambda + (ad-bc) = 0$

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

clearly, a, b, c, d all can effect λ
and therefore type and stability of cp

can small perturbations to the numbers affect behavior?

are there bifurcating behaviors?

example

$$\frac{dx}{dt} = -x + \varepsilon y$$ epsilon (small number)

$$\frac{dy}{dt} = x - y$$

$$\vec{x}' = \begin{bmatrix} -1 & \varepsilon \\ 1 & -1 \end{bmatrix} \vec{x}$$

cp: (0,0)

$$\begin{vmatrix} -1-\lambda & \varepsilon \\ 1 & -1-\lambda \end{vmatrix} = 0$$

$$(1+\lambda)^2 - \varepsilon = 0$$

$$(1+\lambda)^2 = \varepsilon$$

$1+\lambda = \sqrt{\varepsilon}$ or $1+\lambda = -\sqrt{\varepsilon}$

$$\lambda = -1 \pm \sqrt{\varepsilon}$$

if $\varepsilon = 0$, $\lambda = -1, -1$

asymptotically stable

proper or improper node

complete matrix

defective matrix

if $\varepsilon > 0$ (small pos. number)

$$\lambda = -1 \pm \sqrt{\text{small pos.}} = -1 + \sqrt{\varepsilon}\,,\ -1 - \sqrt{\varepsilon}$$

both still negative, AS improper node

if $\varepsilon < 0$ (small neg. #)

$$\lambda = -1 \pm \sqrt{\varepsilon} = -1 \pm i\sqrt{|\varepsilon|} \qquad \text{AS spiral}$$

<u>example</u>

$$\vec{x}\,' = \begin{bmatrix} 1+\varepsilon & -5 \\ 1 & -1 \end{bmatrix} \vec{x} \qquad \text{cp: } (0,0)$$

$$\begin{vmatrix} 1+\varepsilon-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

$$\vdots$$

$$\lambda = \frac{\varepsilon \pm \sqrt{\varepsilon^2 - 4(4-\varepsilon)}}{2}$$

$\varepsilon = 0$: $\quad \lambda = \frac{\sqrt{-16}}{2} = \pm 2i \qquad$ stable, center

ε = small pos. # : $\lambda = \dfrac{\varepsilon \pm \sqrt{\approx -16}}{2} = \overbrace{\dfrac{\varepsilon}{2}}^{>0} \pm \dfrac{\sqrt{\approx -16}}{2} \approx \dfrac{\varepsilon}{2} \pm 2i$

unstable spiral

ε = small neg. # : $\lambda = \dfrac{\varepsilon \pm \sqrt{\approx -16}}{2} \approx \overbrace{\dfrac{\varepsilon}{2}}^{<0} \pm 2i$

asympt. stable spiral

(very slowly spiraling in)