Heat Equation

$u_t = k u_{xx} \qquad 0<x<L \qquad t>0$

solve space problem

BC's : $u(0,t) = u(L,t) = 0$ non-insulated

$u_x(0,t) = u_x(L,t) = 0$ insulated

IC : $u(x,0) = f(x)$ → solve time problem

$$u(x,t) = \underbrace{X(x)}^{\text{space}} \; \underbrace{T(t)}^{\text{time}}$$

$u_t = XT' \qquad u_{xx} = X''T$

$XT' = kX''T$ Heat eq.

$$\frac{X''}{X} = \frac{T'}{kT} = c$$

(in derivation we used $-\lambda$) $\lambda > 0$

separation constant is negative. Why?

$$\boxed{X'' - cX = 0}$$

BC : $u(0,t) = 0 \qquad u(L,t) = 0$

$X(0)\underbrace{T(t)} = 0 \qquad X(L)\underbrace{T(t)} = 0$

generally not zero for all t

$$\boxed{X(0) = 0 \qquad X(L) = 0}$$

can $C=0$ and still satisfy BC's?

if $C=0$, $X''=0 \rightarrow X(x)=Ax+B$ $\quad X(0)=X(L)=0$

$$\left.\begin{array}{l} 0=A(0)+B \rightarrow B=0 \\ 0=A(L) \rightarrow A=0 \end{array}\right\} X=0$$

(trivial solution)

if $X=0$, $u(x,t)=X(x)T(t)=0 \rightarrow$ not what we want

(trivial solution)

can C be positive? $(C>0)$

$X''-CX=0 \rightarrow X(x)=Ae^{\sqrt{C}x}+Be^{-\sqrt{C}x}$ $\quad X(0)=X(L)=0$

$0=A+B \rightarrow B=-A$

$0=Ae^{\sqrt{C}L}+Be^{-\sqrt{C}L}$

$=Ae^{\sqrt{C}L}-Ae^{-\sqrt{C}L}$

$=A\underbrace{(e^{\sqrt{C}L}-e^{-\sqrt{C}L})}_{\substack{\neq 0 \text{ if } L\neq 0 \\ \text{if } C\neq 0}}=0 \rightarrow A=0$

$B=0$

trivial again

So that is why the ~~sp~~ separation constant is negative $(-\lambda,\ \lambda > 0)$

$X'' + \lambda X = 0$

BC's: $X(0) = X(L) = 0$ non-insulated

$X'(0) = X'(L) = 0$ insulated

$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \rightarrow X' = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}x)$

$X(0) = X(L) = 0$

$0 = A \rightarrow X(x) = B\sin(\sqrt{\lambda}x)$

$0 = B\sin(\sqrt{\lambda}L)$ if $B = 0 \rightarrow$ trivial, don't want

so, must force $B \neq 0$

$\sin(\sqrt{\lambda}L) = 0 \rightarrow \sqrt{\lambda}L = n\pi$ because $\sin(n\pi) = 0$

$\sqrt{\lambda} = \frac{n\pi}{L}$

$\lambda = \frac{n^2\pi^2}{L^2}$ for $n = 1, 2, 3, 4, \ldots$

eigenvalues $\lambda_n = \frac{n^2\pi^2}{L^2}$

solution for $X(x) = B\sin\left(\frac{n\pi x}{L}\right)$ for unknown B

but these are multiples of

$\sin\left(\frac{n\pi x}{L}\right) \to$ eigenfunctions $X_n = \sin\left(\frac{n\pi x}{L}\right)$

for insulated, $X_n = \cos\left(\frac{n\pi x}{L}\right)$

Wave: $u_{tt} = a^2 u_{xx}$ $\quad 0 < x < L \quad t > 0$

BC's: $u(0,t) = u(L,t) = 0 \to$ fix positions
(equivalent to non insulated)

$u_x(0,t) = u_x(L,t) = 0 \to$ fix velocities
(similar but not equivalent to insulated)

IC's: $u(x,0) = f(x)$ displacement

$u_t(x,0) = g(x)$ velocity

there are two IC's because u_{tt} results in 2nd order ODE in T after separation $\to$ two conditions
(in heat eq. w/ just $u_t \to$ 1st order)